Generalized Orthogonal Series for Natural Tensor Product Interpolation

by

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Abstract

This 326 page doctoral thesis studies the problem of interpolation by a **natural tensor product (NTP)**, and does so at two contrasting levels of generality: over concrete function spaces and over abstract vector spaces. Whereas prior work uses a Lagrange paradigm to express the NTP interpolant in closed form as a vector-matrix-vector product, the thesis uses a Newton paradigm to express the interpolant in series form. This series is generated by a simple iterative algorithm; the next term is obtained by splitting the previous remainder into a rank-one tensor product using an original, nonlinear, **abstract splitting operator**. The series form is much better suited than the traditional closed form for applications of interpolation to the approximation of a given function or vector by a NTP. (See, for example, Bateman's method for solving Fredholm linear integral equations.)

In the tradition of Bruno Buchberger, the author has named the above series expansions **Geddes series** to honor his thesis supervisor, Professor Keith O. Geddes. Geddes series are orthogonal series expansions in the sense of **generalized inner product (GIP) spaces**. The theory of GIP spaces used here is purely algebraic and is original to the author. Note that all classical inner product spaces are also GIP spaces. In addition, the Geddes series includes all classical orthogonal eigenfunction expansions of Hilbert-Schmidt kernels of Fredholm linear integral operators as special cases.

The thesis studies two concrete interpolation problems in depth: (1) ordinary interpolation of scalar-valued functions of two general variables on the lines of a two-dimensional grid; (2) Taylor interpolation of real-valued functions of two real variables on the two rectilinear coordinate lines through a specified point. For problem (2), the thesis develops remainder formulas in both integral and mean-value forms; the thesis also gives explicit, rigorous, pointwise and uniform error estimates, as well as a sufficient condition for the uniform convergence of infinite Geddes series solutions over compact rectangles. The thesis uses these new infinite series techniques to provide an original derivation and proof of the well-known Neumann addition formula for the order-zero Bessel function of the first kind.

The thesis develops the Geddes series solution of problem (2) as a special case of the **dual asymptotic expansion (DAE)**. The author originally introduced DAEs in his master's thesis, *Theory and Applications of Dual Asymptotic Expansions*. Building on this earlier work, the doctoral thesis classifies DAEs into **weak, strong, and l'Hôpital types**. It develops the uniqueness theory of weak DAEs using an original **asymptotic splitting operator**. It also develops the existence theory of strong DAEs using original **covariant and contravariant asymptotic composition operators**, and gives an algorithm for deciding the question of existence. These asymptotic splitting and asymptotic composition operators provide a fundamentally new way to separate the variables of bivariate functions in applications.

In order to show that DAEs are also orthogonal series over GIP spaces, the thesis develops the **asymptotic inner product space**, which is the quintessential example of a nonclassical GIP space. As a by-product, the thesis provides a completely new foundation for classical asymptotic analysis in one real variable.

The thesis uses the theory of DAEs to develop three new algorithms for the automatic derivation and proof of **tensor product identities** (such as the binomial theorem and the addition formulas for cosine and sine). The proof phase of these algorithms is based on an original **homogeneous hyperbolic eigenproblem** whose unique solution is proven to be the zero function. The thesis also implements one of these algorithms as a complete derivation and proof system in Version 8 of the Maple® computer algebra system. The entire Maple code is included in an appendix, along with some illustrative examples. (Maple is a registered trademark of Waterloo Maple Inc.)

The thesis uses the formal methods of theoretical computer algebra to develop three sets of **normal forms for tensor product expressions**, four **idempotent normal functions**, and five **invariants for deciding the equivalence of expressions** over the tensor product of two abstract vector spaces. This work solves the zero equivalence problem for tensor products, and is based both on the author's **DIRECT algorithm** (see www.acronymfinder.com) for reducing expressions to normal form and on the author's **generalized dual spaces**, which unify the classical algebraic and topological notions of dual spaces by abstracting the essential properties of both.

This interdisciplinary thesis is written for a broad mathematical audience and is largely self-contained. The results of the thesis will be of interest to researchers in multivariate interpolation and approximation theory, asymptotic analysis, computer algebra, and numerical analysis.

Acknowledgements

Although I am indeed the sole author of this thesis, I am by no means the sole contributor! So many people have contributed to my thesis, to my education, and to my life, and it is now my great pleasure to take this opportunity to thank them.

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I thank all the members of the Symbolic Computation Group, the Ontario Research Centre for Computer Algebra, and the employees of Maplesoft for contributing to a research environment that has proven immensely rewarding. You have taught me what it means to belong to a community of scholars. I especially thank **Jacques Carette** and **Eugene Zima** for valuable constructive criticism and for forcing me to think on my feet—it's been both enlightening *and* entertaining! I also thank **Rob Corless** and **Azar Shakoori** for their steady and sincere encouragement over the years—it is indeed a pleasure to know you. Additional thanks go to **David Linder**, **Edgardo Cheb-Terrab**, and **Bruno Salvy** for making extremely helpful contributions to my thesis via the MapleDev e-mailing list. Last, but certainly not least, I thank my colleague and long-time friend **Colin Campbell** for his expertise with mathematical software of all kinds. Your generous nature and home-spun hospitality are deeply appreciated by this stranger in a strange land!

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While at Liberty High School, I was fortunate to learn calculus from **Doris Helms**. I thank you for encouraging me to pursue independent research even at this early age, and for fostering my love of mathematics for its own sake. I still refer to the copy of Angus Taylor's book on *Advanced Calculus* which you gave me. While at Joliet West High School, I was fortunate to learn trigonometry from **Steven Szabo**. It was a privilege to be in your class and to have access to a mathematician with a doctorate—you showed me *mathematics the way it was meant to be done*, and whetted my appetite for more. Thank you, Dr. Szabo, for launching me on the journey of a lifetime—like Robert Frost, I too have taken the road "less traveled by, and that has made all the difference."

Now that I have recounted the entire history of my mathematical upbringing, I have the pleasure of acknowledging those who have provided me with the personal foundation which has been so indispensable to my professional life. First and foremost, I thank my mother **Katherine** and my step-father **Lou** for providing the bedrock of support on which my career is built and the fortress of never-ending concern which has enabled me to weather all the storms of graduate student life. Your patience and understanding with the relentless demands of graduate study will surely earn you a place among the Saints of the Ages! I also gratefully acknowledge the continuing influence of my father **Robert**, and I thank you for the great gift of music that you have given me—it is an enduring legacy which I will always treasure. I want to reassure my brother **Paul** that I have not forgotten him. Some day, we will surely saddle up and ride together towards happier trails.

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Dedication

I dedicate this thesis with love to the one who peers into this pool of thought and in the stillness, sees her own soul reflected back to her:

> I met and knew you, and then
> I grew and met the new you.
> The you I knew was one of two:
> One was undone; one was begun.

How can I ever repay the debt of gratitude I owe to you? From the beginning, you made my very life possible; you shaped my career even before it began; and then—as if you had not been generous enough for one lifetime—you allowed me to see the deepest secrets of the universe unfold in you like the petals of a red, red rose.

> And fare thee weel, my only Luve,
>   And fare thee weel a while!
> And I will come again, my Luve,
>   Tho' it were ten thousand mile.
>
> —Robert Burns

Contents

List of Figures

Chapter 1

A New Program of Research

This thesis has been written with two distinct goals in mind: The short-term goal of the thesis is, of course, to present a substantial body of original research to satisfy the requirements of the doctoral program in which the author is enrolled; however, the long-term goal of the thesis is even more ambitious—for the thesis aspires to create an enduring foundation for research which will sustain both the author and his future students throughout their academic careers. The thesis has therefore been written mindful of two distinct bodies of original research results: the smaller body of results which the limitations of time and space permitted to be included in the thesis proper, and the larger body of results which comprise the author's ongoing program of mathematical research. It became necessary to make such a distinction when the preliminary topics which the author originally anticipated could be dispatched in the *first* chapter of the thesis later turned into the first *nine* chapters of the thesis over the course of eight months of full-time effort.

Given this explosion of mathematical detail, it is clear that providing a complete exposition of the larger body of results would substantially increase the size of the thesis. This would certainly place an unfair burden on both the reader and author alike. The author therefore accepts that it is both necessary and desirable to set some limits on the scope of the thesis; however, the author also deems it necessary to place the results of this thesis into a broader context, and that is the primary purpose of this chapter.

We shall begin this chapter with a personal retrospective which describes how the author's principal research methods originated and evolved to their present state. While waxing nostalgic, we shall dispense with the details in order to cover a lot of ground quickly; the background that is assumed of the reader here will be provided in more detail later in the thesis. As we progress, we shall highlight the distinguishing features of the author's research methods and point out the principal advantages of these methods over conventional alternatives.

Next, we shall discuss the organization of the thesis itself, summarizing the contents of each chapter and explaining how the principal results of each chapter fit together in the context of the thesis as a whole. We shall follow this overview with a brief discussion of the theoretical, applied, and computational perspectives that blend together to form the threefold mathematical philosophy of the thesis; we shall also consider some timely comments on algorithmic mathematics from Bruno Buchberger. We shall end this chapter by assessing the state of the art in high-performance mathematical computation on the desktop computer; by accurately uncovering the facts and carefully considering their implications, we will ensure that we can successfully realize the computational ambitions of the thesis.

1.1 A Personal Retrospective

This is the story of a young man, a nonlinear operator, and a dream. The young man is the author of this doctoral thesis—naturally! The nonlinear operator is the author's very own *abstract splitting operator*. In a typical application, this operator transforms a continuous function of two variables into a rank-one natural tensor product with specified interpolation properties on a compact rectangular domain. Repeated iterations of the splitting operator construct natural tensor product interpolants with arbitrarily high rank and arbitrarily small approximation error, and infinitely many iterations generate a uniformly convergent infinite series expansion for the original function. The dream is to draw the attention of the mathematical community to the many practical and intriguing uses of this splitting operator in the fields of multivariate interpolation and approximation, applied multivariate analysis, and symbolic and numerical computation. To that end, this thesis unveils the splitting operator as the beating heart of a growing body of mathematical techniques. These techniques are applicable to a wide variety of problems of genuine interest, and yet are extremely simple to carry out; it is the author's fond hope that this happy marriage of mathematical versatility with mathematical simplicity will yield many healthy offspring in the course of time.

1.1.1 The Drama Unfolds

The year is 1998. Continuing from the last episode's exciting cliffhanger, we watch in breathless anticipation while our protagonist—having extricated himself from his difficulties—writes his whopping 178 page master's thesis: *Theory and Applications of Dual Asymptotic Expansions* [Cha98]. Peeking over his shoulder, we learn that dual asymptotic expansions are a new kind of series expansion for functions of two real variables; the definition proposed by our young man, now at a slightly younger age, is equivalent to the following: The dual asymptotic expansion of $f(x, y)$ to n terms as $x \to x_0$ or $y \to y_0$ is a rank-n natural[1] tensor product

$$s_n(x, y) := \sum_{i=0}^{n-1} g_i(x)\, h_i(y)$$

which serves as a univariate asymptotic expansion of $f(x, y)$ *both* as $x \to x_0$ for all fixed y *and* as $y \to y_0$ for all fixed x. We assume that $f : X \times Y \to \mathbb{R}$ for some subsets $X, Y \subset \mathbb{R}$, and we assume that x_0 and y_0 are limit points of X and Y in the topology of the extended real numbers.

Our suddenly younger man, a mere child in the flower of his youth (HA!) soon realizes that whenever the dual asymptotic expansion of $f(x, y)$ as $x \to x_0$ or $y \to y_0$ exists, its terms are *uniquely determined* by the function f and the point (x_0, y_0). The man-child is thus lead to construct an idempotent, multiplicative, nonlinear operator $\Upsilon_{(x_0,y_0)}$ which allows us to calculate the next term $\Upsilon_{(x_0,y_0)}\, r_i$ of the dual asymptotic expansion from the previous remainder r_i. More precisely, by setting $s_0 := 0$ and $r_0 := f$, and iterating

$$s_{i+1} := s_i + \Upsilon_{(x_0,y_0)}\, r_i \quad \text{and} \quad r_{i+1} := f - s_{i+1} \quad \text{for} \quad i = 0, 1, \ldots, n-1, \qquad (1.1)$$

we obtain s_n, the n-term dual asymptotic expansion of f at (x_0, y_0). We call $\Upsilon_{(x_0,y_0)}$ the *asymptotic* splitting operator at (x_0, y_0).

Much to his surprise, our young man discovers that the operator $\Upsilon_{(x_0,y_0)}$ has some very interesting interpolation properties. For example, whenever (x_0, y_0) lies in the domain of f, the rank-one natural tensor product $\Upsilon_{(x_0,y_0)} f(x, y)$ interpolates $f(x, y)$ on the rectilinear coordinate lines $x = x_0$ and $y = y_0$. Here is something even more interesting: Whenever f has continuous partial derivatives of order $2n$ in an open neighborhood of (x_0, y_0), the

[1] We will clarify the meaning of the term "natural" later in this chapter after we have introduced the necessary background. For the time being, it suffices to think of "natural" as a synonym for "not necessarily polynomial."

dual asymptotic expansion $s_n(x, y)$ performs *Taylor interpolation* of order $n-1$ for the expression $f(x, y)$ on the lines $x = x_0$ and $y = y_0$. This means that $s_n(x, y)$ interpolates both $f(x, y)$ and its *normal derivatives* up to order $n-1$ on $x = x_0$ and $y = y_0$. In addition, the remainder r_n satisfies

$$r_n(x, y) = O\left((x - x_0)^n (y - y_0)^n\right)$$

in any compact subneighborhood of (x_0, y_0). Since $s_n(x, y)$ is *by definition* a natural tensor product, the operator $\Upsilon_{(x_0,y_0)}$ becomes the beating heart of a simple iteration (1.1) that performs Taylor interpolation on the lines $x = x_0$ and $y = y_0$ using natural tensor products!

1.1.2 The Saga Continues

While aging gracefully, it occurs to our young man that other kinds of interpolation properties can be achieved using a similar iteration based on the *same operator* $\Upsilon_{(x_0,y_0)}$ if we simply choose *different parameters* (x_0, y_0). For example, suppose we want to construct a rank-n natural tensor product $s_n(x, y)$ which interpolates $f(x, y)$ on the $2n$ distinct lines

$$x = x_0,\ x = x_1,\ \ldots,\ x = x_{n-1} \quad \text{and} \quad y = y_0,\ y = y_1,\ \ldots,\ y = y_{n-1}$$

of a suitable $n \times n$ grid. If we simply replace (x_0, y_0) by (x_i, y_i) in the original iteration (1.1), we obtain a more general iteration

$$s_{i+1} := s_i + \Upsilon_{(x_i,y_i)}\, r_i \quad \text{and} \quad r_{i+1} := f - s_{i+1} \quad \text{for} \quad i = 0, 1, \ldots, n-1 \tag{1.2}$$

which constructs a natural tensor product s_n with precisely the specified interpolation properties! In addition, if f has continuous partial derivatives of order $2n$ in an open neighborhood of the grid, the remainder r_n satisfies

$$r_n(x, y) = O\left(\prod_{i=0}^{n-1}(x - x_i) \prod_{j=0}^{n-1}(y - y_j)\right)$$

in any compact subneighborhood of the grid.

Note that when we take *distinct* lines, the above scheme performs *ordinary* interpolation on the lines of a two-dimensional grid using a rank-n natural tensor product. If we *repeat* each line exactly once, the above scheme now performs *simple Hermite interpolation* on the lines of the *same* two-dimensional grid using a rank-$2n$ natural tensor product. This means that the resulting natural tensor product s_{2n} has twice as many terms, and interpolates *both* the original function f *and* its first-order normal derivatives on the lines of the original grid. If we repeat each line *with its own arbitrary multiplicity*, the above scheme now performs *full Hermite interpolation* on the lines of the original grid. This means that the resulting natural tensor product interpolates the function f and its normal derivatives on each grid line up to an order determined by the multiplicity of that line. Note that full Hermite interpolation is a very general scheme, and includes *all* the interpolation schemes mentioned thus far as *special cases*. For example, if we repeat the lines $x = x_0$ and $y = y_0$ each n times and perform full Hermite interpolation, we recover the *original* Taylor interpolation scheme of order $n - 1$ on the lines $x = x_0$ and $y = y_0$.

While full Hermite interpolation might appear to be the pinnacle of generality amongst interpolation schemes, it is merely a lofty plateau on a much larger mountain. While climbing up this mountain, our somewhat older and by now hyperventilating young man finds out that we can reformulate the concrete interpolation properties we have studied thus far in a more abstract way using linear functionals. For example, the property of interpolation on the lines $x = x_0$ and $y = y_0$ can be expressed in terms of the linear functionals

$$\phi_{x_0}(u) := u(x_0) \quad \text{and} \quad \psi_{y_0}(v) := v(y_0);$$

these linear functionals operate on univariate functions u and v, and have well-defined extensions[2] to linear operators on bivariate functions such as f and r_i. As an added benefit, this abstract approach allows us to specify extremely *general* interpolation properties using *arbitrary* linear functionals ϕ and ψ.

Between wheezes, our exhausted young man also learns that we do not need to interpolate *functions* f at all, but can instead interpolate *linear functional data* derived from *vectors* w in an abstract vector space W. Upon collapsing at the top of the mountain, our young man finally realizes, just before losing consciousness, that given vector spaces U and V whose tensor product satisfies $U \otimes V \subset W$, a particular vector $w \in W$, and suitable linear functionals $\phi \in U^*$ and $\psi \in V^*$, we can construct an idempotent, homogeneous, nonlinear operator $\Omega_{(\phi,\psi)}$ that transforms w into a rank-one natural tensor product in $U \otimes V$, all the while interpolating the data derived from w by the specified linear functionals ϕ and ψ.

We conclude that truth is indeed stranger than fiction, for that, dear reader, is the story of how the *abstract* splitting operator $\Omega_{(\phi,\psi)}$ came to be—through the loving labors

[2]Extending linear functionals on univariate functions to linear operators on bivariate functions is an important technical detail which we shall address carefully later in the thesis; for the time being, however, we suppress this detail in order to avoid complicating this introduction.

of an originally younger but now somewhat older young man-child in the gracefully aging flower of his semi-conscious youth!

1.1.3 The View from the Mountaintop

Aside from the rarefied mountain air, crystal-clear mountain streams, and all the trail mix one can eat, what are the benefits of ascending to this breathtaking, awe-inspiring, downright stupefying level of abstraction?[3] Given a vector $w \in W$ and suitable families $\{\phi_i\}_{i=0}^{n-1} \subset U^*$ and $\{\psi_i\}_{i=0}^{n-1} \subset V^*$ of linear functionals, we can modify the previous iteration (1.2) by replacing the *asymptotic* splitting operator $\Upsilon_{(x_i,y_i)}$ and the *function* f by the *abstract* splitting operator $\Omega_{(\phi_i,\psi_i)}$ and the *vector* w to obtain the even more general iteration

$$s_{i+1} := s_i + \Omega_{(\phi_i,\psi_i)}\, r_i \quad \text{and} \quad r_{i+1} := w - s_{i+1} \quad \text{for} \quad i = 0, 1, \ldots, n-1 \tag{1.3}$$

where $s_0 := 0$ and $r_0 := w$. The resulting iteration constructs a rank-n natural tensor product $s_n \in U \otimes V$ which *interpolates all the data* derived from w by the specified $2n$ linear functionals. In this way, the *abstract* splitting operator $\Omega_{(\phi_i,\psi_i)}$ becomes the beating heart of a new iteration (1.3) that performs natural tensor product interpolation in a much more general setting than the previous iteration (1.2) based on the *asymptotic* splitting operator $\Upsilon_{(x_i,y_i)}$.

This new iteration (1.3) accommodates *all* the interpolation schemes of the previous iteration (1.2), and many new interpolations schemes as well. For example, the new iteration (1.3) allows us to perform *Hermite-Birkhoff interpolation* on the lines of a two-dimensional grid, which further generalizes the full Hermite interpolation scheme by allowing us to *skip* intermediate orders of normal differentiation at will; in particular, Abel-Gontscharoff interpolation (which interpolates the normal derivatives of order *exactly* i on the two corresponding lines $x = x_i$ and $y = y_i$ for $0 \le i \le n-1$) and Lidstone interpolation (which interpolates all the normal derivatives of *even* order $2i$ for $0 \le i \le n-1$ on all four lines $x = x_0$, $x = x_1$ and $y = y_0$, $y = y_1$) are two standard Hermite-Birkhoff interpolation schemes which are *not* Hermite interpolation schemes [Dav63, p. 28, Dover].

The new iteration (1.3) also allows us to perform natural tensor product interpolation on ordinary functions of two variables in a much more general setting. For example, we can now interpolate functions of two *vector* variables rather than two *real* variables

[3] The author apologizes to readers who suffer from severe vertigo—*just don't look* dooooooo...

using essentially the same techniques; hence, this level of abstraction also benefits us by facilitating *multidimensional* interpolation.

There is one more useful level of generality that we shall consider in this thesis: If we take suitable *infinite families* of linear functionals $\{\phi_i\}_{i=0}^{\infty} \subset U^*$ and $\{\psi_i\}_{i=0}^{\infty} \subset V^*$ and apply the new iteration (1.3) for $n = 1, 2, 3, \ldots$, the resulting infinite family of natural tensor products $\{s_n\}_{n=0}^{\infty} \subset U \otimes V$ becomes the sequence of partial sums of an *infinite series for natural tensor product interpolation*. We denote this infinite series by the symbol s_∞, and we call it a *Geddes series*, in honor of Professor Keith O. Geddes, who supervised both the author's master's and doctoral theses. Professor Geddes also cofounded the Maple® computer algebra system, which has been an indispensable tool in the author's mathematical research for many years; indeed, without a tool such as Maple, many of the insights of this thesis would have taken much, much longer to come to light, and some might never have been discovered at all.[4]

By introducing a topology, we can consider questions such as whether the Geddes series s_∞ converges, and if so, whether it converges to the original vector w. Convergence of the series s_∞ to w has important implications for the theory of approximation by natural tensor products: In particular, *convergence guarantees desirable approximation properties* of the natural tensor products s_n that result from truncating s_∞ to n terms, or equivalently, from performing interpolation scheme (1.3) with exactly n iterations.

By deliberate design, the dual asymptotic expansion of a function f to *infinitely many terms* at a point (x_0, y_0) in the domain of an infinitely-differentiable function f is a special case of the Geddes series. Consequently, there is substantial overlap between the methods of this doctoral thesis and the methods of the author's master's thesis [Cha98]; however, the doctoral thesis provides a more complete theoretical framework than this earlier work, including a rigorous remainder theory. We will use the dual asymptotic expansion and asymptotic splitting operator $\Upsilon_{(x_0,y_0)}$ as running examples throughout the doctoral thesis to illustrate the Geddes series and abstract splitting operator $\Omega_{(\phi,\psi)}$.

A more surprising special case of the Geddes series arises from a classical Hilbert space technique: The orthonormal eigenfunction expansion

$$f(x, y) = \sum_{i=0}^{\infty} \frac{u_i(x)\, u_i(y)}{\lambda_i} \tag{1.4}$$

[4] Maple is a registered trademark of Waterloo Maple Inc. The documentation for Maple 8 consists of [Map02a], [Map02b], [MGH+02b], and [MGH+02a], which are listed in order of increasing sophistication; this suite of manuals covers a wide range of topics of interest to beginning, intermediate, and advanced Maple users.

of the symmetric Hilbert-Schmidt kernel $f(x, y)$ of the Fredholm linear integral operator

$$(Fu)(x) := \int_0^1 f(x, y)\, u(y)\, dy \quad \text{for} \quad 0 \le x \le 1$$

is actually a Geddes series with respect to the family of linear functionals

$$\phi_i(u) := \langle u, u_i \rangle := \int_0^1 u(x)\, u_i(x)\, dx \quad \text{for} \quad i = 0, 1, 2, \ldots$$

induced by the real inner product $\langle \bullet, \bullet \rangle$ and the normalized eigenfunctions $\{u_i\}_{i=0}^{\infty}$.

Note that the more abstract approach is sufficiently general to accommodate not only an important special case of the author's *original asymptotic methods*, but this *classical Hilbert space method* as well. *These two seemingly unrelated methods can thus be absorbed into a single unified theoretical framework!* This illustrates another benefit of abstraction: the unification of disparate branches of mathematics.

In applications to linear integral equations, the principal benefit of the eigenfunction expansion (1.4) is that it *separates the variables* of the kernel $f(x, y)$ in a termwise fashion. As a direct consequence, truncating the eigenfunction expansion to n terms approximates the kernel $f(x, y)$ by a rank-n natural tensor product

$$s_n(x, y) := \sum_{i=0}^{n-1} \frac{u_i(x)\, u_i(y)}{\lambda_i}.$$

The kernel approximation $s_n(x, y)$, in turn, leads to a very practical method for approximating the solution u of the linear integral equation $(I - \lambda F)(u) = w$ for suitable values of the parameter λ and a known function w.

There is a substantial practical problem with this classical Hilbert space approach, however: The eigenfunctions $\{u_i\}_{i=0}^{\infty}$ of the integral operator F can be extremely difficult to determine, for each u_i is *by definition* the solution of *another* integral equation $(I - \lambda_i F)(u_i) = 0$. Alas, one *cannot* write down the eigenfunction expansion (1.4) or the kernel approximation $s_n(x, y)$ explicitly *unless* these eigenfunctions are known; thus, the principal drawback of the classical Hilbert space approach to solving the integral equation $(I - \lambda F)(u) = w$ by tensor product approximation of the kernel is that this approach requires us to solve the integral equations $(I - \lambda_i F)(u_i) = 0$ *exactly* for the eigenfunctions u_i. The exact solutions of these auxiliary integral equations cannot, in general, be determined explicitly by a finitary algorithm. Worse yet, we cannot, in general, determine each eigenvalue λ_i *exactly* by a finitary algorithm. In light of these difficulties, what hope do we have of solving the original integral equation by practical means?

Do not despair, dear reader! The author's algorithms for constructing a Geddes series—in the general case—overcome these problems readily. The key is to generate *alternative bases* that can be used *instead of eigenfunctions* to separate the variables of the kernel $f(x, y)$. By deliberate design, the Geddes series for natural tensor product interpolation does precisely that, and does so in a very *natural* way—hence, the name.

1.1.4 Just Do What Comes Naturally!

What exactly do we mean by the term "natural" as applied to tensor products? Naturalness is a *relative concept*, and is defined by the same mathematical objects that specify an interpolation problem. The following simplified formulation of the concept is suitable for most applications: A tensor product $s_n \in U \otimes V$ is said to be "natural" *with respect to a particular vector* $w \in W$ *and suitable families of linear functionals* $\{\phi_i\}_{i=0}^{n-1} \subset U^*$ *and* $\{\psi_i\}_{i=0}^{n-1} \subset V^*$ if the factors in each term of the tensor product

$$s_n := \sum_{i=0}^{n-1} u_i \otimes v_i$$

can be derived from w *using linear combinations of the specified functionals*. Since there are technical details that must be addressed in order make this definition precise, it may clarify matters to consider the following simple example:

Suppose we wish to interpolate the expression $f(x, y)$ on the $2n$ lines

$$x = x_i \quad \text{and} \quad y = y_i \quad \text{for} \quad 0 \le i \le n-1$$

of a suitable $n \times n$ grid. In this context, a tensor product

$$s_n(x, y) := \sum_{i=0}^{n-1} g_i(x)\, h_i(y)$$

is called "natural" if the univariate expressions in each term of $s_n(x, y)$ are linear combinations of the *cross-sections*

$$f(x, y_i) \quad \text{and} \quad f(x_i, y) \quad \text{for} \quad 0 \le i \le n-1$$

of the original expression $f(x, y)$.

Precisely how do natural tensor product techniques generate *alternative bases*, as claimed earlier? This is illustrated by the concrete example given above, in which naturalness has the following intriguing consequence: Whenever we interpolate a *bivariate* expression $f(x, y)$ on the lines of a two-dimensional grid by a natural tensor product $s_n(x, y)$, we always generate *univariate* bases

$$\{f(x, y_i)\}_{i=0}^{n-1} \quad \text{and} \quad \{f(x_i, y)\}_{i=0}^{n-1} \tag{1.5}$$

from a function space *intrinsic to the problem at hand!* This means that when we interpolate a *rational* function of two variables, we generate bases of *rational* functions of one variable. Similarly, *trigonometric* problems generate *trigonometric* bases, *exponential* problems generate *exponential* bases, and so on.

The use of *natural bases* (1.5) that are easily derived from the original expression $f(x, y)$ makes natural tensor product techniques *adaptive* in a way that goes well beyond the usual meaning of the term. This *ultra-adaptive* approach to tensor product interpolation can offer both qualitative and quantitative advantages over more conventional approaches that use the same *fixed bases* for all expressions $f(x, y)$. The author has found concrete examples which illustrate how the natural bases (1.5) are adept at capturing the qualitative behavior of functions f with steep slopes, nonuniform oscillations, and even integrable singularities. *The extraordinary ability to adapt to the given function f is one of the principal advantages of natural tensor product techniques.*

Recall that the historical goal of interpolation has been to approximate a given function by a *simpler* function which reproduces selected values (or some other linear functional data) of the original. In the example given above, how is the natural tensor product $s_n(x, y)$ simpler than the original expression $f(x, y)$? It is simpler in two distinct ways:

1. The natural tensor product $s_n(x, y)$ approximates the values of $f(x, y)$ anywhere inside a rectangular region using only the values of $f(x, y)$ on the lines of the specified two-dimensional grid. This reduces the evaluation of $f(x, y)$ over a two-dimensional set to the evaluation of $f(x, y)$ on a one-dimensional subset, *thereby cutting the dimension of the evaluation problem in half.*

2. The natural tensor product $s_n(x, y)$ *separates the variables* of the expression $f(x, y)$ in a termwise fashion, which changes the fundamental algebraic structure of the expression in a way that is highly advantageous in applications. For example, separation

of variables reduces a double integral over a rectangular region to a collection of independent single integrals, *thereby cutting the dimension of the integration problem in half.*

In both instances, approximating $f(x, y)$ by the natural tensor product $s_n(x, y)$ *reduces the dimension* of the problem at hand. Thus, we can think of natural tensor product interpolation as an effective method for the *dimensional reduction* of a multidimensional problem.

Please note that the sense in which $s_n(x, y)$ is "simpler" than $f(x, y)$ does *not* include the notion "less costly to evaluate." The cost of evaluating the natural bases (1.5) is *exactly* the same as the cost of evaluating $f(x, y)$ itself![5] In order to anticipate possible objections to this point, the author states emphatically that the primary goal of natural tensor product interpolation is *not* to reduce the cost of evaluating the given expression $f(x, y)$; on the contrary, we *assume* that we are already given $f(x, y)$ in some explicit form suitable for computation. *The primary goal of natural tensor product interpolation is to separate the variables x and y in the context of applications!*

The author has successfully applied natural tensor product interpolation and the separation of variables to develop effective problem-solving techniques for a variety of applications, including the automatic derivation and proof of tensor product identities, the accurate numerical evaluation of multiple integrals, and the approximate solution of linear Fredholm integral equations. Applications such as these provide ample reason to "just do what comes naturally!"

1.2 Chapter Markers along the Thesis Highway

This thesis represents a substantial journey for both reader and author alike. Along the way, it will help to have some markers with which to gauge our progress; to that end, the thesis is organized into chapters, whose numbers and titles are listed below.[6] It will also help to acquaint ourselves in advance with the major attractions that we will encounter along the way; that is why this section provides a detailed travel guide which describes

[5] It is true, however, that reducing the evaluation of $f(x, y)$ over a rectangular region to the evaluation of $f(x, y)$ over a *finite* collection of lines *prepares the way* for a reduction in computational cost. If necessary, an "optimized" tensor product can be produced by "post-processing" the natural tensor product $s_n(x, y)$ using conventional univariate interpolation techniques. We shall return to this point in Subsection 1.4.1.

[6] The thesis is also organized into *sections* and *subsections*. The author has made a careful distinction between these two kinds of subdivisions throughout the thesis; the reader is encouraged to do likewise in order to avoid becoming confused while navigating these subdivisions of the thesis.

each succeeding chapter's major points of interest. This travel guide provides an informal introduction to the main results of the thesis, and explains how these results relate to one another in the context of the thesis as a whole.

Overview of Chapter 2

The foundations of this thesis are constructed primarily from three general areas of mathematics: abstract linear algebra, functional analysis, and asymptotic analysis. We will devote an entire chapter of the thesis to each of these three foundational areas. *Foundations from Abstract Linear Algebra* is the first of these foundational chapters. This chapter begins with a brief survey of useful notations from set theory, focusing especially on cardinal arithmetic. This will enable us to define vector space dimensions quite generally as cardinal numbers, and to manipulate these dimensions in a meaningful way even for infinite-dimensional vector spaces. Next, we will quickly review the basic facts about vector spaces, such as the existence of Hamel bases, and will then develop some specialized algebraic formalisms for function spaces. After this, we will specialize further by developing function algebras. We will develop polynomial algebras as an important special case which will prove to be a fertile source of examples throughout the thesis. We will conclude this chapter with a brief introduction to tensor products in the concrete setting of function spaces, and will include illustrative examples based on polynomial algebras.

Overview of Chapter 3

Foundations from Functional Analysis is the second of the three foundational chapters of the thesis. This chapter begins by discussing the topological spaces of central importance to the thesis. The extended real numbers are introduced as the natural topological setting for asymptotic analysis in one real variable. Dual spaces play an important role in the thesis, and are developed here in a novel way; we will develop the fundamental properties of both algebraic and topological dual spaces, and then unify these properties to create the notion of a generalized dual space. We will use the generalized dual space throughout the thesis to develop dual-purpose results which are independent of any particular notion of the dual space. This chapter also introduces all the normed linear spaces of real-valued functions which the thesis will require. In addition, this chapter develops two purely algebraic formalisms with numerous applications to analysis: the cross-sections of bivariate functions and the parametric extensions of linear operators to spaces of bivariate functions.

In the remainder of this chapter, we will use these formalisms to develop familiar examples of important linear operators, such as point-evaluation operators, and ordinary, partial, and normal derivative operators.

Overview of Chapter 4

Tensor Product Normal Forms and Invariants builds on the brief introduction to the tensor products of concrete function spaces in Chapter 2 by developing the basic theory of tensor products of abstract vector spaces. This chapter celebrates the numerous applications of the universal property of tensor products and repeatedly demonstrates that this abstract property is a labor-saving device. The chapter also shows that tensor products in the sense originally defined for function spaces possess the universal property, and are therefore tensor products in the sense of abstract vector spaces. In addition, the chapter develops formal proof techniques for exploiting duality between the factors of a tensor product space as an additional labor-saving device.

The main focus of this chapter is actually a systematic and algorithmic study of the problem of nonuniqueness of tensor product representations. This chapter uses the methods of theoretical computer algebra to develop a complete and original solution to this problem. The author's solution develops three different kinds of normal forms, four different idempotent normal functions, and five different invariants for tensor product expressions. One of the most useful consequences of this work is a simple criterion for determining the rank of a tensor product based on linear independence; we will use this criterion later in the thesis to show that the n-th partial sum of a dual asymptotic expansion is automatically a tensor product of rank n, for example.

Overview of Chapter 5

This thesis treats two bivariate interpolation problems in careful detail: ordinary interpolation on the distinct lines of a two-dimensional grid, and Taylor interpolation on two rectilinear coordinate lines; as we previously noted, these two problems represent the opposite ends of the spectrum of possibilities for full Hermite interpolation on the lines of a two-dimensional grid. *Applications to Interpolation on Grid Lines* uses the results of the previous three chapters to the develop the algebraic theory relevant to the first of these two bivariate interpolation problems. The chapter begins by introducing an important special case of the asymptotic splitting operator. After developing the fundamental properties of the asymptotic splitting operator in this special case, the chapter uses the operator to develop an iterative algorithm which performs natural tensor product interpolation on the lines of a two-dimensional grid. The chapter concludes by revisiting the special case of the asymptotic splitting operator in order to abstract its structure, thereby hinting at the nature of the more general abstract splitting operator that is to come.

Overview of Chapter 6

Foundations from Asymptotic Analysis is the third of the three foundational chapters of the thesis. This chapter not only completes the foundations of the thesis—it also prepares the way for a fundamentally new foundation for asymptotic analysis in one real variable. This new foundation is based on the author's own asymptotic inner product spaces, which are quintessential examples of generalized inner product spaces. We will discuss generalized inner product spaces and their applications to asymptotic analysis in Chapter 11.

Chapter 6 begins with a traditional look at the well-known asymptotic order relations which provide the customary foundation for classical asymptotic analysis. This is followed by a much deeper, nontraditional look at the most important asymptotic order relation, thereby stimulating the development of new classes of functions in order to provide suitable hypotheses for rigorous work in asymptotic analysis. After this, the chapter develops an original asymptotic order relation which plays an important role in both old and new foundations of asymptotic analysis. The chapter concludes by developing the basic facts about asymptotic sequences, asymptotic series, and asymptotic expansions from both traditional and nontraditional points of view.

Overview of Chapter 7

The Theory of Dual Asymptotic Expansions develops a new approach to asymptotic analysis for real-valued functions of two real variables. These original asymptotic methods are based on the author's own dual asymptotic expansions and asymptotic splitting operator; in fact, this chapter presents a vastly new-and-improved approach which both simplifies and extends the original asymptotic methods first presented in the author's master's thesis [Cha98].

The instruments of this new-and-improved theory are the following three original constructions: rectilinear limits, which are a new kind of limit for functions of two real variables; and covariant and contravariant asymptotic composition operators, which compose two given bivariate functions in order to produce a new bivariate function. This chapter uses these three original constructions to define and study the asymptotic splitting operator in full generality—a study which culminates in a complete characterization of the structure of the operator. The chapter also introduces a simple classification scheme for dual asymptotic expansions by articulating a collection of weak hypotheses and a collection of strong hypotheses which are well-suited to the development of subsequent theory. The chapter concludes by using the structure theory of the asymptotic splitting operator to develop a definitive uniqueness theory for dual asymptotic expansions under the weak hypotheses, and a definitive existence theory for dual asymptotic expansions under the strong hypotheses. We will also develop an algorithm for deciding the question of existence under the strong hypotheses.

Overview of Chapter 8

Applications to Taylor Interpolation treats the second of the two major bivariate interpolation problems of the thesis. In this chapter, we will study applications of asymptotic analysis to the Taylor interpolation of real-valued functions of one or two real variables. We will see that asymptotic expansions in one real variable can be characterized by their Taylor interpolation properties at a point on the real line, and that dual asymptotic expansions in two real variables can be characterized by their Taylor interpolation properties on two rectilinear coordinate lines in the plane.

In both one and two real variables, we will use the multiplicities of zero points and zero lines to develop simple hypotheses which will have far-reaching consequences for the local algebraic and asymptotic behavior of sequences of functions. These hypotheses will also allow us to exploit l'Hôpital's rule in a systematic way, thereby reducing the evaluation of limits to the computation and evaluation of derivatives. We will use these hypotheses to develop important subclasses of both univariate asymptotic expansions and weak dual asymptotic expansions, and will name these new subclasses after l'Hôpital.

In addition, we will develop rigorous remainder theories for Taylor interpolation in both one and two real variables. These remainder theories, which we will develop in both integral and mean-value forms, will yield practical error estimates as well as convergence criteria for l'Hôpital asymptotic expansions with infinitely many terms. We will conclude this chapter by using these two-variable methods to give an original derivation and proof of a well-known uniformly convergent infinite series expansion involving Bessel functions.

Overview of Chapter 9

Algorithms for Deriving and Proving Identities develops three original algorithms for the automatic derivation and proof of tensor product identities; in so doing, this chapter introduces a new, complementary approach to research in this general area of computer algebra. Each of the three algorithms of this chapter uses the same two-phase design: a derivation phase followed by a proof phase. The derivation phase uses the theories of l'Hôpital, strong, and weak dual asymptotic expansions developed in the previous two chapters to generate a tensor product expression from a given closed-form expression in two variables. The proof phase uses an original uniqueness result for the homogeneous hyperbolic eigenproblem to show that the closed-form expression and the tensor product expression are identically equal. This chapter includes detailed examples which not only illustrate how to carry out these new derivation and proof techniques, but also demonstrate that these techniques can be applied to a wide variety of identities involving exponential, logarithm, trigonometric, and polynomial functions.

Overview of Chapter 10

The next three chapters establish an array of important connections: connections between the mathematical literature and the original results of the thesis; connections between the two different bivariate interpolation problems studied in depth by the thesis; connections between seemingly unrelated branches of mathematics such as linear algebra, asymptotic analysis, and interpolation theory; and connections between the author's past research accomplishments and future research plans. As we draw near to the end of our mathematical journey, we will tie all of these various and sundry things together.

Natural Tensor Product Interpolation develops abstractions which both unify and generalize the two fundamental interpolation problems of the thesis. Along the way, this chapter also codifies many standard and original results in the fundamental theory of natural tensor product interpolation—doing so at an unprecedented level of generality! The methods of this chapter draw heavily upon both concrete matrix algebra and abstract linear algebra. On the concrete side, we will develop formulas for the determinant and inverse of matrices partitioned into blocks. On the abstract side, we will develop a precise original notion of commuting linear functionals over linear configurations of vector spaces, and we will use this notion to define the *abstract* splitting operator in full generality. We will then extend this notion to *families* of commuting linear functions in order to specify both weak and strong interpolation problems. We will also give the mathematical literature's first rigorous definition of the space of *natural* tensor products. We will conclude this chapter by using our partitioned matrix formulas to develop an iterative algorithm which uses the abstract splitting operator to generate exact series solutions to the strong interpolation problem in the space of natural tensor products.

Overview of Chapter 11

Generalized Orthogonal Series Expansions develops a new, purely algebraic approach to generalized inner products on vector spaces over an arbitrary field. The author developed this original approach to meet the diverse needs of applications taken from all three traditional branches of mathematics—geometry, analysis, and algebra. Indeed, the author's algebraic axioms for generalized inner product spaces have evolved over the course of nearly two decades of mathematical research in these areas. This slow, patient, application-driven evolution has distilled the essence of these generalized inner product spaces down to two simple axioms whose power and applicability should not be underestimated. In this chapter of the thesis, we will develop these axioms and describe their applications to our exact series solutions to natural tensor product interpolation problems.

Overview of Chapter 12

Reflections and Plans ends the thesis as we began—with a nostalgic retrospective—but this time, with the benefit of details. After summarizing the main results of the thesis and reflecting upon where we have been, we will plan a new journey.

Are We There Yet?

Yes! The end of our journey will be heralded with Appendix A, *Using Maple to Derive and Prove Identities*. This appendix supplements Chapter 9, *Algorithms for Deriving and Proving Identities*, and includes a brief, thoroughly-documented Maple code, along with a number of illustrative examples carried out with the help of Maple.

We conclude this travel guide by noting that we will occasionally need to shift gears during our journey in order to work at an appropriate level of abstraction. We will shift into low gear to navigate the rugged terrain of detailed examples in concrete function spaces, and shift into high gear to enjoy the smooth and speedy progress facilitated by general theoretical results over abstract vector spaces.

1.3 The Philosophical Engine under the Mathematical Hood

Now that we have planned our mathematical journey, let us take a moment to inspect the philosophical engine that powers our mathematical progress. There are, of course, many forces that drive contemporary mathematics forward. In the author's experience, the dynamic interplay between theory, application, and computation is an irresistible force of attraction for fresh insights and new discoveries. For this reason, the author readily embraces the theoretical, applied, and computational points of view in mathematics, and strives to blend these three perspectives together into a single research philosophy. *In keeping with this philosophy, the thesis aspires to demonstrate that natural tensor product techniques are based on a solid theoretical foundation, have many interesting applications, and are very well-suited for computation.*

1.3.1 Experimental Mathematics: A New Research Paradigm

In practicing his threefold philosophy, the author has found it natural to follow a simple threefold path when conducting much of his mathematical research. This new research paradigm, which is both widely applicable and increasingly popular among contemporary mathematicians, consists of the following three steps:

1. Use a computer algebra system such as Maple to perform computational experiments on specific examples.

2. Use empirical observations based on these experimental results to formulate a mathematical conjecture and devise a feasible proof strategy.

3. Use rigorous mathematical methods to prove the conjecture and produce a new mathematical theorem.

On more than one occasion, traveling this threefold path has lead the author through a complete cycle of mathematical discovery—from experimental observation to proven theorem! For example, this approach lead to the discovery and proof of a new (non-classical) Geddes series $s_\infty(x, y)$ that converges uniformly to the piecewise linear function $(x, y) \mapsto |x - y|$ on the unit square $[0, 1] \times [0, 1]$. *This example is especially instructive because it provided the first rigorous proof of uniform convergence for a member of this new class of infinite series expansions.* Furthermore, the proof techniques used in this example can undoubtedly be adapted for other piecewise polynomial functions; hence, this example paves the way for further progress in the study of convergence.

1.3.2 Algorithmic Mathematics: A New Research Standard

In his recent article, *Computer Algebra: The End of Mathematics?*, Bruno Buchberger[7] acknowledges computer algebra systems as useful tools for mathematical experimentation leading to insight and the advancement of mathematical theory [Buc02]. Buchberger goes on to argue that computer algebra has an even greater role to play in the advancement of mathematics: Computer algebra provides the impetus for a program Buchberger calls "the algorithmization of mathematics"—the development of mathematical problem-solving techniques based on finitary algorithms rather than nonconstructive methods like quantifying over an infinite set or invoking the axiom of choice. Because it is more difficult to develop mathematics along algorithmic lines, the algorithmization of mathematics necessarily leads to a deepening of mathematical theory. Buchberger argues that the successful automation of algorithmic mathematics in a computer algebra system does *not* indicate that mathematics has been trivialized; quite to the contrary, it is proof positive that mathematical theory has reached even greater depths! From this point of view, algorithmic mathematics becomes the crowning achievement of mathematical theory, and automation becomes the crowning achievement of algorithmic mathematics.

The author embraces Buchberger's point of view that algorithmic methods are strongly preferable to nonconstructive methods, and that automation is both the ultimate goal and the ultimate test of any algorithm. That is why the thesis develops *constructive* methods for tensor product interpolation and approximation. That is why the thesis develops a remainder theory that is not only rigorous, but *algorithmic* as well—with the potential for a computer-algebra implementation that *automatically* yields rigorous error bounds for all functions from a well-defined class, such as the class of holonomic functions.[8] That is why the author's research program works to develop *automatic* methods for the evaluation of multiple integrals and *automatic* methods for the solution of linear integral equations,

[7]Buchberger invented the Gröbner basis for multivariate polynomial ideals. Buchberger's algorithm for constructing Gröbner bases is among the most fundamental algorithms in computational algebra, and has numerous important applications in commutative algebra and algebraic geometry. For example, Buchberger's algorithm can be used to explicitly solve a system of multivariate polynomial equations—an application of great interest to scientists and engineers as well as mathematicians!

[8]A multivariate function is called holonomic if its partial derivatives of all orders span a finite-dimensional vector space over the field of multivariate rational functions. The algebra of holonomic functions includes many of the elementary functions and many of the special functions of mathematical physics.

and provides *constructive*, rigorous error estimates for both problems. Indeed, *most* of the mathematics of the author's research program is shaped in a significant way by the author's philosophical preference for algorithmic mathematics with a strong potential for successful automation.

1.4 Superscalar Spark Plugs and Pipelined Fuel Injectors

In keeping with the mathematical philosophy of the thesis, we shall strive to develop our mathematics along *algorithmic* lines and to implement the resulting algorithms in an *automated* fashion. This implementation work can be carried out on almost any platform, from a stand-alone desktop computer to a networked cluster of computers working in parallel, or even on a high-end parallel/vector supercomputer.[9]

1.4.1 What Is the Cost of Computing with Natural Bases?

Is a single modern desktop computer powerful enough to implement algorithms for natural tensor product interpolation and approximation? To answer this question, we must understand the significant computational implications of the concept of *naturalness*, which we defined informally in Subsection 1.1.4 and illustrated with a concrete example in the same subsection; in this example, we noted that natural tensor product interpolation of an expression $f(x, y)$ on the lines of a two-dimensional grid generates *natural bases*

$$\{f(x, y_i)\}_{i=0}^{n-1} \quad \text{and} \quad \{f(x_i, y)\}_{i=0}^{n-1}$$

which displace the *fixed bases* of conventional interpolation methods. The resulting natural bases may consist of arbitrary univariate functions, and thus may be more expensive to evaluate than conventional bases consisting of polynomials or rational functions, for example.

[9] In the interest of good relations between academia and industry, the author hereby acknowledges the ownership of the trademarks of all the commercial computer hardware and software products and technologies that will be introduced in this section: Intel, Pentium, and MMX are registered trademarks of Intel Corp.; Advanced Micro Devices, AMD Athlon, and 3DNow! are registered trademarks of Advanced Micro Devices; Microsoft and Windows are registered trademarks of Microsoft Corp.; UNIX is a registered trademark of The Open Group; and Mathematica is a registered trademark of Wolfram Research Inc.

We can now reformulate our original question as follows: Can the state of the computational art on the desktop computer support the increased computational demands of natural bases? If computational cost is of paramount concern, there is nothing to prevent us from *interpolating the natural bases themselves* using polynomials or rational functions. Just as natural tensor product interpolation reduces the evaluation of $f(x, y)$ to the evaluation of a finite collection of natural basis functions, post-processing by conventional univariate interpolation reduces the cost of evaluating the natural bases to the cost of evaluating polynomials or rational functions.

This two-stage approach will indeed construct a tensor product that has been optimized for numerical evaluation—at the price of introducing additional approximation error, which we can of course control with additional effort. In general, however, is it really necessary to use bivariate and univariate interpolation techniques in combination, or are natural tensor product techniques computationally feasible in their own right?

Historically, the answer to questions of feasibility has depended on the computational primitives at one's disposal. Let us define a "computational primitive" on a digital computer to be an operation that is either implemented directly in hardware or can be implemented in software with a relatively small number of machine instructions. For example, in the early days of digital computing, addition and subtraction were implemented directly in hardware, while multiplication and division were frequently implemented in software using hardware primitives consisting primarily of additions, subtractions, and shift operations. Thus, by our working definition, the computational primitives on early digital computers consisted of the four basic arithmetic operations.

Long before the invention of the electronic computer, however, early interpolation methods relied on bases of polynomials and rational functions, presumably because they can be evaluated using only simple arithmetic. For precisely this reason, interpolation by polynomials and rational functions was particularly well-suited for implementation on early digital computers.

1.4.2 Developments in Computer Hardware and Software

The contrast between the early days of digital computing and the present day is striking: Today's desktop computers process complex video and audio signals in real time for popular multimedia applications, render high-resolution graphics images for interactive games, and compress and decompress data streams at high rates of speed for transmission over communications networks. Multimedia applications, interactive games, and network communications constitute such important shares of the computer market—and are so immensely computationally intensive—that they have literally become the driving force in the evolution of desktop computer architecture at both the hardware and software levels. As a happy consequence, computational mathematicians everywhere are reaping substantial benefits from these trends in popular entertainment!

Consider the following developments in *desktop computer hardware* which have occurred in the Intel® line of Pentium® processors:

1. Every Pentium processor contains a floating-point unit (FPU) which implements *all four arithmetic operations* for floating-point numbers in single, double, *and* extended precisions. In addition, the FPU implements approximations of *transcendental constants* such as π, $\log_2 e$, and $\log_2 10$, as well as the *algebraic function* $\sqrt{x}$, and *transcendental functions* like $2^x - 1$ and $y \cdot \log_2(1 + x)$, as well as $\sin(x)$, $\cos(x)$, $\tan(x)$, and $\arctan(x)$, for floating-point arguments x and y. By deliberate design, *all of the elementary transcendental functions*—namely b^x and $\log_b(x)$ for any base $b > 0$, as well as all the trigonometric and hyperbolic functions and their inverses—*can be expressed in terms of these hardware primitives via simple algebra*, often with fewer than a dozen machine instructions. With this level of hardware support, we may consider the Pentium's computational primitives to include *all of the elementary transcendental functions as well as the four basic arithmetic operations!*

2. The architecture of the Intel line of Pentium processors includes many sophisticated performance-enhancing features: Integrating the main processor, the FPU, and cache memory onto a single chip that uses wider data paths and faster clock speeds than ever before is merely the beginning! The Pentium line also implements various kinds of parallelism, such as a *superscalar architecture*[10] with *pipelined execution*,[11] and *array processing based on the single-instruction-multiple-data (SIMD) paradigm*.[12] On the Pentium line, the machine instructions that perform array processing are called *matrix math extensions* (the correct name, abbreviated MMX®) or *multimedia extensions* (the popular but incorrect name, also abbreviated MMX®) and *streaming*

[10] A processor with a superscalar architecture can execute more than one machine instruction simultaneously on scalar data when circumstances are suitable.

[11] Pipelined execution allows a processor to overlap the execution of machine instructions—the next instruction begins executing before the previous instruction has finished. In suitable circumstances, this substantially increases the rate at which instructions can be performed.

[12] A SIMD array processor can execute a single machine instruction on an entire array of data simultaneously—for example, calculating the square root of each entry of a vector or matrix of floating-point numbers all at once.

SIMD extensions (abbreviated SSE, with later extensions referred to as SSE2). Note that many other manufacturers are incorporating similar array-processing instructions into their own lines of processors (e.g., The AMD Athlon® by Advanced Micro Devices® has 3DNow!® technology).

Consider also the accompanying developments in *desktop computer software*, which build on the hardware developments described above, benefiting both applications developers and end-users alike:

1. Intel provides optimized *performance libraries* to give developers an easy way to exploit the high-performance features of Intel hardware in their own applications. In the author's opinion, the level of mathematics supported by these libraries is quite exciting. For example:

   (a) Intel's *integrated performance primitives* provide optimized functions for multimedia applications involving *signal processing, image processing, and computer vision.* This library includes functions for the fast Fourier transform (FFT), a wavelet transform, a variety of filters, and much more.

   (b) Intel's *math kernel library* provides optimized functions for mathematical, scientific, engineering, and financial applications. This library includes optimized versions of two standard *numerical linear algebra* libraries, LAPACK and BLAS.[13] Intel's math kernel library also includes functions for one- and two-dimensional FFTs, as well as the *vector math library*, which is a collection of elementary algebraic and transcendental functions that operate *elementwise* on vectors of floating-point numbers.

2. Intel also provides support for software development in high-level programming languages with its *optimizing C++ and FORTRAN compilers.* These compilers enable programmers to benefit from the high-performance features of Intel hardware with minimal effort, and support *parallel execution* in a number of ways, *including the use of Intel's MMX, SSE, and SSE2 array-processing instructions*, the automatic vectorization of code, and the automatic parallelization of code for execution on symmetric multiprocessing systems.

[13] LAPACK stands for "linear algebra package" and BLAS stands for "basic linear algebra subprograms." LAPACK is a high-level library built on the low-level library of BLAS. Both LAPACK and BLAS are high-quality, transportable software, freely available online (http://www.netlib.org).

3. The research team of Whaley and Dongarra of the University of Tennessee at Knoxville have an ongoing project called ATLAS, which stands for "automatically tuned linear algebra software" [WD98]. The goal of ATLAS is to create portable linear algebra software that achieves *optimal performance* on a wide range of modern computing platforms.[14] The current version of ATLAS supports the complete BLAS and a subset of LAPACK.

4. Maple (as of Version 6) achieves high performance in numerical linear algebra computations on Microsoft® Windows® platforms *by using Intel's math kernel library.* Maple (as of Version 7) achieves high performance for similar computations on UNIX® platforms *by using the ATLAS library.*

1.4.3 Trends in Mathematical Computing on the Desktop

Taken all together, these developments reveal a significant trend in the evolution of computer hardware and software *for mathematical computation on desktop computers:* Vendors like Intel have incorporated high-performance features like array-processing instructions into their processors and provided both optimized libraries and optimizing compilers to enable applications developers to reap the benefits of this enhanced hardware. In turn, both academic and commercial software developers have used these hardware enhancements and software development tools to implement mathematical software packages for important applications like signal processing, image processing, and numerical linear algebra.

The way that modern hardware and vendor-supplied software libraries *are actually being used by professional developers of mathematical software* suggests that we need to extend our concept of computational primitives considerably. A computational primitive—as defined by common practice among professional developers—is *any* standard mathematical operation or function with an implementation that is efficient enough for use as a fundamental building-block in higher-level applications. In this extended sense, the FFT is a widely-accepted computational primitive in signal and image processing applications, and the BLAS are widely-accepted computational primitives for numerical linear algebra.

The hardware and software developments described thus far pertain largely to *numerical computation.* The state of the art in *symbolic computation* has made great strides

[14] For example, ATLAS has implementations that support SSE, SSE2, and 3DNow! array-processing instructions for optimal performance on platforms with Intel and Advanced Micro Devices processors. ATLAS is freely available online (http://math-atlas.sourceforge.net).

as well. For example, every modern *computer algebra system* can represent and manipulate all the elementary transcendental functions as *symbolic objects*, and has detailed knowledge of the special values, identities, derivatives, and antiderivatives of these functions. Thus, developments in *both numerical and symbolic computing* strongly support the view that modern-day computational primitives include the elementary transcendental functions. Some computer algebra systems, such as Maple® and Mathematica®, go well beyond this by implementing extensive libraries of *the special functions of mathematical physics*, and possess a detailed knowledge of their many symbolic and numerical properties. Nowadays, the wide availability of such advanced implementations on such powerful machines enables us to use almost *any* standard mathematical function as a computational primitive in the extended sense!

All things considered, we conclude that modern desktop computers provide a vastly expanded collection of computational primitives and an unprecedented level of raw computing power which can easily support the computational demands of natural tensor product techniques. Thanks to remarkable strides in the development of computer technology, the time for natural tensor product methods to come into their own has arrived at last.

Chapter 2

Foundations from Abstract Linear Algebra

In the mathematical diners scattered along the thesis highway, abstract linear algebra is served in two distinct flavors: with and without topology. Plain linear algebra which contains no topological ingredients of any kind is sufficient to develop the formalisms of interpolation theory in Chapter 5. Spicy linear algebra which is seasoned liberally with metrics and norms and inner products—a subject better known as functional analysis—is necessary to obtain the error estimates and convergence criteria in Chapter 8. In this chapter and the next, we will sample some simple but tasty dishes from both sides of the menu.

This chapter summarizes a variety of standard results from abstract linear algebra. We will begin with a brief survey of useful notations from set theory, focusing especially on cardinal arithmetic. This will enable us to define vector space dimensions quite generally as cardinal numbers, and to manipulate these dimensions in a meaningful way even for infinite-dimensional vector spaces. Next, we will quickly review the basic facts about vector spaces, such as the existence of Hamel bases, and will then develop some specialized algebraic formalisms for function spaces. After this, we will specialize further by developing function algebras. We will develop polynomial algebras as an important special case which will prove to be a fertile source of examples throughout the thesis. We will conclude this chapter with a brief introduction to tensor products in the concrete setting of function spaces, and will include illustrative examples based on polynomial algebras.

2.1 Number Systems and Cardinal Arithmetic

Let $\mathbb{N} := \{0, 1, 2, \ldots\}$ denote the set of natural numbers. We will denote the set of integers by $\mathbb{Z}$, the set of rational numbers by $\mathbb{Q}$, the set of real numbers by $\mathbb{R}$, and the set of complex numbers by $\mathbb{C}$.

The following material on cardinal arithmetic can be found in [Kam50], [Sho67], and [HJ78]. To every well-defined set A, we assign a cardinal number, denoted $|A|$, which we call the **cardinality of** A. If A is an infinite set, the cardinal number $|A|$ is said to be **transfinite**. We denote the smallest transfinite cardinal by $\aleph_0$. Here are the cardinalities of some familiar sets:

$$\begin{aligned} |\{0, 1, \ldots, n-1\}| &= n \quad \text{for all} \quad n \in \mathbb{N}, \\ |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| &= \aleph_0, \\ |\mathbb{R}| = |\mathbb{C}| &= 2^{\aleph_0}. \end{aligned}$$

Note that some of the infinite sets listed above have the same cardinality. We write $|A| = |B|$ whenever the sets A and B are in one-to-one correspondence. We write $|A| \leq |B|$ whenever A is in one-to-one correspondence with a subset of B, and we write $|A| < |B|$ whenever $|A| \leq |B|$ and $|A| \neq |B|$. The **Cantor-Schröder-Bernstein theorem** asserts that for all well-defined sets A and B, the conditions $|A| \leq |B|$ and $|B| \leq |A|$ imply $|A| = |B|$.

Given a function $f : A \to B$, we call A the **domain of** f and write $\operatorname{dom} f := A$. If $S \subset A$, we call $f(S) := \{f(s) : s \in S\}$ the **image of** S **under** f. In particular, we call $f(A)$ the **range of** f and write $\operatorname{ran} f := f(A)$. If $T \subset B$, we call $f^{-1}(T) := \{a \in A : f(a) \in T\}$ the **inverse image of** T **under** f. If $b \in B$, we will denote $f^{-1}(\{b\})$ by $f^{-1}(b)$ for convenience.

We define the **setwise exponential**, denoted B^A, to be the set of all functions from A into B; hence, $f \in B^A$ if and only if $f : A \to B$. In particular, we define $B^\varnothing := \{\varnothing\}$ for all sets B, but define $\varnothing^A := \varnothing$ whenever $A \neq \varnothing$. Note that $\{\varnothing\}$ is a *different* set than $\varnothing$ because $|\{\varnothing\}| = 1$, but $|\varnothing| = 0$.

We can calculate the cardinalities of the disjoint union, Cartesian product, and setwise exponential of the sets A and B by taking the sum, product, and exponential of the cardinal numbers $|A|$ and $|B|$, as follows:

$$\begin{aligned} |A \cup B| &= |A| + |B| \quad \text{when} \quad A \cap B = \varnothing, \\ |A \times B| &= |A| \cdot |B|, \\ \left|B^A\right| &= |B|^{|A|}. \end{aligned}$$

Let $n \in \mathbb{N}$. Assume $n \geq 1$ and $B \neq \varnothing$. We denote the n-fold Cartesian product of B with itself by B^n, and write $b \in B^n$ as $b = (b_0, b_{1,} \ldots, b_{n-1})$. Let $A := \{0, 1, \ldots, n-1\}$, and note that every $b \in B^n$ induces a function $\hat{b} \in B^A$ via $\hat{b}(a) := b_a$. The canonical map $b \mapsto \hat{b}$ places B^n and B^A in one-to-one correspondence; hence, $|B^n| = \left|B^A\right|$. Let us confirm this with a direct calculation that illustrates the principles of cardinal arithmetic:

$$|B^n| = \Big|\underbrace{B \times B \times \cdots \times B}_{n \text{ times}}\Big| = \underbrace{|B| \cdot |B| \cdots |B|}_{n \text{ times}} = |B|^n = |B|^{|A|} = \left|B^A\right|.$$

If A is any well-defined set, the set of all subsets of A is called the **power set of** A, and is denoted by 2^A; hence, $B \in 2^A$ if and only if $B \subset A$. If $A \neq \varnothing$ and $B \in 2^A$, we can define a function $\chi_B \in \{0,1\}^A$ by letting

$$\chi_B(a) := \begin{cases} 1 & \text{if } a \in B \\ 0 & \text{if } a \notin B \end{cases}$$

for all $a \in A$. We call χ_B the **characteristic function of** B. Since the canonical map $B \mapsto \chi_B$ places 2^A and $\{0,1\}^A$ in one-to-one correspondence, we can calculate the cardinal number of subsets of A by

$$\left|2^A\right| = \left|\{0,1\}^A\right| = |\{0,1\}|^{|A|} = 2^{|A|}.$$

The formula $\left|2^A\right| = 2^{|A|}$ is also valid when $A = \varnothing$, because $\varnothing$ has exactly one subset, namely itself.

The **Cantor diagonal theorem** asserts that $|A| < 2^{|A|}$ for all well-defined sets A. For example,

$$|\mathbb{N}| = \aleph_0 < 2^{\aleph_0} = |\mathbb{R}|,$$

which says that the real numbers are uncountable.

We will find the set-theoretic constructions of this section especially useful as we develop notions related to abstract vector spaces in general and concrete functions spaces in particular. This is the focus of the next section.

2.2 Vector Spaces and Function Spaces

In this section, V will denote a vector space over an arbitrary field F. We say that $B \subset V$ is a **Hamel basis of** V if every element of V can be written as a *finite* linear combination of elements of B in a *unique* way. Equivalently, B is a Hamel basis if the elements of B are *linearly independent* and *generate* V. If $S \subset G \subset V$, where S is a linearly independent set and G generates V, Lang in [Lan65, pp. 85–86] assures us that V has a Hamel basis B with $S \subset B \subset G$. For arbitrary vector spaces V, the construction of B uses Zorn's lemma, which is equivalent to the axiom of choice. An immediate corollary of this construction is that every nontrivial vector space $V \neq \{0\}$ has a Hamel basis.

If B_1 and B_2 are any two Hamel bases of $V \neq \{0\}$, Lang in [Lan65, pp. 86–87] assures us that $|B_1| = |B_2|$; consequently, we may define the **dimension of** V **over** F to be the cardinal number $|B|$, where B is *any* Hamel basis for V. We write $\dim_F V := |B|$. Since the trivial space $V = \{0\}$ has no Hamel basis, we define $\dim_F V := 0$ in that case.

Let $B := \{v_i \in V : i \in I\}$ be a Hamel basis of $V \neq \{0\}$, and note that B is indexed by some set I. The linear independence of B implies the map $I \to B$ given by $i \mapsto v_i$ is injective. The map is surjective by definition since I indexes B; hence, I and B are in one-to-one correspondence, and we have

$$\dim_F V := |B| = |I|.$$

Example 2.1 (Univariate Polynomials) *Let $F[x]$ denote the set of all polynomials whose coefficients lie in the field F and whose indeterminate is x. The set $F[x]$ is an infinite-dimensional vector space over F under the usual addition of polynomials and multiplication of polynomials by scalars. The set of monomials $\{x^n : n \in \mathbb{N}\}$ is a Hamel basis for $F[x]$, and the dimension of $F[x]$ over F satisfies*

$$\dim_F F[x] := |\{x^n : n \in \mathbb{N}\}| = |\mathbb{N}| = \aleph_0.$$

If $S \neq \varnothing$ is an arbitrary set, the set V^S of all functions from S into V becomes a vector space over F when addition and scalar multiplication are defined by

$$\begin{aligned}(f+g)(s) &:= f(s)+g(s)\\(\alpha f)(s) &:= \alpha\, f(s)\end{aligned}$$

for all $f, g \in V^S$, $\alpha \in F$, and $s \in S$. A **function space over** F is any proper or improper subspace of a vector space of the form V^S for some set S and some vector space V over the field F.

Let W be a vector space over F. Following the conventions of algebra exemplified in [Lan65, p. 76], [Jac85, p. 169], and [AW92, p. 109], we denote the set of all vector space homomorphisms from V into W by $\mathrm{Hom}_F(V, W)$. This set, which consists of all F-linear functions from V into W, is a subspace of the function space W^V defined above; hence, $\mathrm{Hom}_F(V, W)$ is a function space over F. If $\{v_i \in V : i \in I\}$ is a Hamel basis of V and $\{w_i \in W : i \in I\}$ is an arbitrary subset of W indexed by I, Lang in [Lan65, pp. 84–85] assures us that we can define an F-linear function $T : V \to W$ uniquely by prescribing $Tv_i := w_i$ for all $i \in I$.

In the next two subsections, we will develop two different ways to construct function spaces: We can use vector spaces of polynomials to *induce* function spaces by interpreting polynomials as functions, and we can form bivariate function spaces from univariate function spaces via *tensor products*.

2.2.1 Function Algebras and Polynomial Algebras

In this subsection, we will build upon the basic structure of a vector space to develop the notion of an algebra. We will then use algebras to construct new examples of function spaces.

Let F denote an arbitrary field. We call A an **algebra over** F if A is a vector space over F and A has an associative, bilinear multiplication which maps $A \times A \to A$; we also call A an F**-algebra**. If the vector space structure of A is that of a function space over F, we call A a **function algebra over** F.

Since F is a one-dimensional vector space over itself, replacing V by F in the function space V^S yields the function space F^S; this function space over F becomes a function algebra over F under **pointwise multiplication**, which is defined by

$$(fg)(s) := f(s)\, g(s)$$

for all $f, g \in F^S$ and $s \in S$. Pointwise multiplication is not only associative and bilinear, but commutative as well; the constant function 1 is the multiplicative identity. If $f \in F^S$ satisfies $f(s) \neq 0$ for all $s \in S$, the multiplicative inverse of f, denoted by $1/f$, is defined by

$$\left(\frac{1}{f}\right)(s) := \frac{1}{f(s)} \quad \text{for all} \quad s \in S.$$

The vector space $F[x]$ of polynomials in x is an algebra over F under the usual multiplication of polynomials. Replacing the arbitrary set S by the field F in the function algebra F^S yields the function algebra F^F. What is the relationship between the algebra $F[x]$ of *polynomials* and the algebra F^F of *functions?* We shall devote the remainder of this subsection to this question.

Let the polynomial $p(x) \in F[x]$ be given by the algebraic expression

$$p(x) := \sum_{i=0}^{n} \alpha_i \, x^i$$

with formal degree $n \in \mathbb{N}$ and coefficients $\{\alpha_i\}_{i=0}^n \subset F$. Given any scalar $\xi \in F$, define the scalar value $p(\xi) \in F$ by

$$p(\xi) := \sum_{i=0}^{n} \alpha_i \, \xi^i.$$

Let the solitary symbol p denote the **induced function** from $F \to F$ given by $\xi \mapsto p(\xi)$ for all $\xi \in F$. Every polynomial $p(x) \in F[x]$ induces a function $p \in F^F$ by this construction. In fact, the map from $F[x] \to F^F$ given by $p(x) \mapsto p$ is an algebra homomorphism; consequently, under the substitution $x := \xi$, the usual scaling, addition, and multiplication of *polynomials* are compatible with the scaling, addition, and pointwise multiplication of *functions* as defined above.

Example 2.2 (Univariate Polynomials) *If $p_1(x) := x$ and $p_2(x) := x^2$, then for all $\xi \in F$, we have*

$$(p_1(x)\, p_2(x))\,|_{x:=\xi} = \left(x \cdot x^2\right)|_{x:=\xi} = x^3|_{x:=\xi} = \xi^3 = \xi \cdot \xi^2 = p_1(\xi)\, p_2(\xi) =: (p_1\, p_2)(\xi),$$

which means that the function induced by the product of the polynomials $p_1(x)$ and $p_2(x)$ is identical to the pointwise product of the induced functions p_1 and p_2.

Note that it is possible for *distinct polynomials* in $F[x]$ to induce the *same function* in F^F. For example, although $p_1(x) := x$ and $p_2(x) := x^2$ are distinct polynomials, the induced functions $p_1 := (\xi \mapsto \xi)$ and $p_2 := (\xi \mapsto \xi^2)$ are identical over the finite field $F := \{0, 1\} = \mathbb{Z}/2\mathbb{Z}$ since every element of F is idempotent. This phenomenon cannot occur when F is a subfield of $\mathbb{C}$, such as $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$ itself; in that case, the homomorphism $p(x) \mapsto p$ is injective, and we may safely think of the polynomial algebra $F[x]$ as a subalgebra of the function algebra F^F by identifying the polynomial $p(x)$ with the induced function p.

Similarly, let $F[x, y]$ denotes the F-algebra of bivariate polynomials generated by distinct indeterminates x and y. Given the polynomial $p(x, y) \in F[x, y]$ and the point $(\xi, \eta) \in F \times F$, define the scalar value $p(\xi, \eta) \in F$ in a manner analogous to the univariate case. We will let the solitary symbol p denote the **induced function** from $F \times F \to F$ given by $(\xi, \eta) \mapsto p(\xi, \eta)$ for all $(\xi, \eta) \in F \times F$. Whenever $F \subset \mathbb{C}$ is a subfield, the map from $F[x, y] \to F^{F \times F}$ given by $p(x, y) \mapsto p$ is an injective algebra homomorphism; in that case, we may think of the polynomial algebra $F[x, y]$ as a subalgebra of the function algebra $F^{F \times F}$ by identifying the polynomial $p(x, y)$ with the induced function p.

Polynomial algebras will be especially useful for developing examples to illustrate various ideas in tensor product theory. We will take our first formal glimpse of tensor product theory in the next subsection.

2.2.2 Tensor Products of Function Spaces

In this subsection, we will introduce algebraic tensor products over concrete function spaces and state the basic properties of tensor products, most without proof. In Chapter 4, we will revisit algebraic tensor products in the more general setting of abstract vector spaces; the additional abstraction will provide us with the tools we will need to prove the basic properties of tensor products quite easily. We begin with the following fundamental definition:

Definition 2.3 *Let X and Y be arbitrary sets, and let F be an arbitrary field. Let $U \subset F^X$ and $V \subset F^Y$ be function spaces, and let $W := F^{X \times Y}$. Given any functions $u \in U$ and $v \in V$, we can define a new function $u \otimes v \in W$ by*

$$(u \otimes v)(x, y) := u(x)\, v(y) \quad \textit{for all} \quad (x, y) \in X \times Y.$$

*We call the function $u \otimes v$ the **tensor product of the functions** u **and** v.*

We can define a subspace $U \otimes V \subset W$ by

$$U \otimes V := \operatorname{span}_F G \quad \textit{where} \quad G := \{u \otimes v : u \in U,\ v \in V\}.$$

We call the function space $U \otimes V$ the ***tensor product of the function spaces*** U ***and*** V*. The elements of $U \otimes V$ are called* ***tensor products****, and the elements of the set G, which generate the space $U \otimes V$, are called* ***dyads*** *or* ***elementary tensor products****.*

Please take note of the following remark:

Remark 2.4 *The map $U \times V \to U \otimes V$ given by $(u, v) \mapsto u \otimes v$ is bilinear. This property is of fundamental importance to tensor product theory.*

The following illustrative example uses the polynomial algebras $F[x]$ and $F[y]$ in a very careful way to construct the tensor product of the *function spaces* induced by $F[x]$ and $F[y]$:

Example 2.5 (Polynomial Algebras) *Let $F \subset \mathbb{C}$ be a subfield. Recall that under this assumption, we can identify the polynomial algebras $F[x]$ and $F[y]$ with subspaces of the function space F^F via the maps $p(x) \mapsto p$ and $q(y) \mapsto q$, where $p(x)$ and $q(y)$ denote polynomials in $F[x]$ and $F[y]$, and p and q denote the corresponding induced functions in F^F. With these identifications, the construction $F[x] \otimes F[y]$ becomes a well-defined subspace of the function space $F^{F \times F}$. Recall also that we can identify the polynomial algebra $F[x, y]$ with a subspace of the function space $F^{F \times F}$ when $F \subset \mathbb{C}$ is a subfield. Assuming these identifications, what is the relationship between the two subspaces $F[x] \otimes F[y]$ and $F[x, y]$ of the function space $F^{F \times F}$?*

Let us identify the bivariate function $p \otimes q$ with the bivariate polynomial $p(x)\, q(y)$. This is justified since

$$(p \otimes q)\,(\xi, \eta) := p(\xi)\, q(\eta) =: (p(x)\, q(y))\,|_{(x,y):=(\xi,\eta)}$$

for all $(\xi, \eta) \in F \times F$; in words, the tensor product $p \otimes q$ of the induced functions p and q is the same bivariate function induced by the bivariate polynomial $p(x)\, q(y)$. Define the monomials $p_m(x) := x^m$ and $q_n(y) := y^n$ for all $m, n \in \mathbb{N}$, and note that the special case $p := p_m$ and $q := q_n$ allows us to identify the bivariate monomial $x^m y^n$ with the bivariate function $p_m \otimes q_n$. With these identifications, we can now answer the question under consideration:

By the definition of $F[x] \otimes F[y]$,

$$\begin{aligned} F[x] \otimes F[y] &:= \operatorname{span}_F \{p \otimes q : p(x) \in F[x],\ q(y) \in F[y]\} \\ &= \operatorname{span}_F \{p(x)\, q(y) : p(x) \in F[x],\ q(y) \in F[y]\} \subset F[x, y]. \end{aligned}$$

By the definition of $F[x,y]$,

$$\begin{aligned} F[x,y] &:= \operatorname{span}_F \{x^m y^n : m,n \in \mathbb{N}\} \\ &= \operatorname{span}_F \{p_m \otimes q_n : m,n \in \mathbb{N}\} \subset F[x] \otimes F[y]. \end{aligned}$$

These two inclusions together yield the following equality of function spaces:

$$F[x] \otimes F[y] = F[x,y].$$

We conclude that under the appropriate identifications, every tensor product constructed from univariate polynomials is actually a bivariate polynomial, and every bivariate polynomial is actually a tensor product constructed from univariate polynomials. For all intents and purposes, these two kinds of algebraic objects are one and the same!

In order to better understand the structure of the tensor product $U \otimes V$ of function spaces U and V, let us compare $U \otimes V$ to the more familiar internal direct sum $U \oplus V$. We can regard U and V as subspaces of W via the identifications $u(x,y) := u(x)$ and $v(x,y) := v(y)$ for all $u \in U$, $v \in V$, and $(x,y) \in X \times Y$. Similarly, we can regard F as a subspace of U, V, and W via the identifications $\alpha(x) = \alpha(y) = \alpha(x,y) := \alpha$ for all $\alpha \in F$ and $(x,y) \in X \times Y$.

Now define the subspace $U + V \subset W$ via

$$U + V := \{u + v : u \in U,\ v \in V\}.$$

Since the functions in U are independent of y and the functions in V are independent of x, the functions in $U \cap V$ are constant, which means that $U \cap V \subset F$. Assume (for the purpose of comparing $U \otimes V$ and $U \oplus V$, but *not* thereafter) that V contains no constant functions except 0. This ensures that $U \cap V = \{0\}$; hence, the sum $U + V$ is direct, and we may write $U \oplus V$ instead of $U + V$.

If $\{u_i\}_{i \in I}$ is a Hamel basis of U and $\{v_j\}_{j \in J}$ is a Hamel basis of V, then $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$ is a Hamel basis of $U \otimes V$, and

$$\dim_F (U \otimes V) = |I \times J| = |I| \cdot |J| = \dim_F U \cdot \dim_F V.$$

In contrast, the disjoint union $\{u_i\}_{i \in I} \cup \{v_j\}_{j \in J}$ is a Hamel basis of $U \oplus V$; consequently,

$$\dim_F (U \oplus V) := |\{u_i\}_{i \in I} \cup \{v_j\}_{j \in J}| = |I| + |J| = \dim_F U + \dim_F V.$$

We conclude that forming the tensor product $U \otimes V$ *multiplies* the dimensions of U and V, whereas forming the direct sum $U \oplus V$ *adds* the dimensions of U and V.

Remark 2.6 *Because we multiply and add vector space dimensions using cardinal arithmetic, we do not need to assume that U and V are finite-dimensional. This illustrates the benefit of developing the theory of abstract vector spaces on the foundation of set theory: We obtain very general results, free of unnecessary restrictions.*

Now that we have used vector space dimensions to understand the structure of the function space $U \otimes V$, we ask a more pointed question: What do the elements of $U \otimes V$ look like? Since the operation $\otimes$ is bilinear, every tensor product $t \in U \otimes V$ can be written as a finite sum of dyads

$$t = \sum_{i=0}^{n-1} u_i \otimes v_i \tag{2.1}$$

for some $\{u_i\}_{i=0}^{n-1} \subset U$ and $\{v_i\}_{i=0}^{n-1} \subset V$. Equivalently, the representation (2.1) can be written as an algebraic expression in x and y,

$$t(x, y) = \sum_{i=0}^{n-1} u_i(x)\, v_i(y). \tag{2.2}$$

We adopt the convention that when $n = 0$, the resulting empty sum (2.1) is interpreted as the tensor product $t = 0$.

Remark 2.7 *The representation (2.1) of a tensor product $t \in U \otimes V$ is far from unique! Every tensor product has numerous distinct representations as a finite sum of dyads. We will study the implications of this nonuniqueness in detail in Chapter 4.*

Even the tensor product zero (which is the zero function in W) has multiple representations in $U \otimes V$. For example, if $F \subset \mathbb{C}$ is a subfield, zero has the following two-parameter family of nontrivial representations in $F[x] \otimes F[y]$:

$$0 \quad = \quad 1 \otimes (-q) \ + \ (1 - 2p) \otimes q \ + \ p \otimes (2q).$$

The parameters p and q denote the functions induced by arbitrary polynomials $p(x) \in F[x]$ and $q(y) \in F[y]$.

The following lemma for tensor products of *function spaces* describes an extremely useful property of representations of the tensor product zero. We will prove this lemma in full generality for tensor products of *abstract vector spaces* in Chapter 4.

Lemma 2.8 (Cancellation) *Let X and Y be arbitrary sets, and let F be an arbitrary field. Let $U \subset F^X$ and $V \subset F^Y$ be function spaces, and let $\{u_i\}_{i=0}^{n-1} \subset U$ and $\{v_i\}_{i=0}^{n-1} \subset V$, where $n \geq 1$. Assume that*

$$\sum_{i=0}^{n-1} u_i \otimes v_i = 0.$$

The following two results hold independently:

1. *If $\{u_i\}_{i=0}^{n-1}$ is a linearly independent set, then all the elements of $\{v_i\}_{i=0}^{n-1}$ are zero.*
2. *If $\{v_i\}_{i=0}^{n-1}$ is a linearly independent set, then all the elements of $\{u_i\}_{i=0}^{n-1}$ are zero.*

Given an arbitrary tensor product t, there is no *upper* bound on the complexity of the representations of t that we can create by adding distinct nontrivial representations of zero to t. There is, however, a *lower* bound on the complexity of the representations of t, which is formally defined as follows:

Definition 2.9 *Fix a tensor product $t \in U \otimes V$, and consider all possible representations of the form*

$$t = \sum_{i=0}^{n-1} u_i \otimes v_i \quad \text{for} \quad \{u_i\}_{i=0}^{n-1} \subset U, \ \{v_i\}_{i=0}^{n-1} \subset V, \ \text{and } n \in \mathbb{N}. \tag{2.3}$$

*If $m \in \mathbb{N}$ is the minimum number of terms among all possible representations of the tensor product t, we call m the **rank of** t and write* $\operatorname{rank} t := m$.

This definition serves double duty since the notion of rank in the tensor product of *abstract vector spaces* is defined in exactly the same way as for the tensor product of *function spaces*. In either case, how do we know that the rank is well-defined? If N denotes the set of all $n \in \mathbb{N}$ for which the tensor product t has an n-term representation of the form (2.3), we know that $N \neq \varnothing$ since t has at least one such representation by the definition of $U \otimes V$. Since the set $\mathbb{N}$ is well-ordered by $\leq$, the nonempty subset $N \subset \mathbb{N}$ has a unique least element $m := \min N$. This shows that $\operatorname{rank} t := m$ is well-defined.

Note that $\operatorname{rank} 0 := 0$ follows from our convention for the empty sum; inversely, every $t \neq 0$ has $\operatorname{rank} t \geq 1$. Whenever a representation (2.3) of a tensor product $t \neq 0$ has exactly $n = \operatorname{rank} t$ terms, it follows immediately that both $\{u_i\}_{i=0}^{n-1}$ and $\{v_i\}_{i=0}^{n-1}$ are linearly independent sets; otherwise, as Cheney argues in [Che86, p. 11], we could invoke a linear dependence and exploit the bilinearity of $\otimes$ to reduce the number of terms in the representation (2.3) by one, thereby contradicting the minimality of $\operatorname{rank} t$.

The converse of this is also true, but is harder to prove: If both $\{u_i\}_{i=0}^{n-1}$ and $\{v_i\}_{i=0}^{n-1}$ are linearly independent sets, then the representation (2.3) yields a tensor product $t \neq 0$ (by Cancellation Lemma 2.8), and the number of terms n is the minimum possible (by an invariance principle that we will discuss in Chapter 4); in symbols, $n = \operatorname{rank} t$.

This converse gives us a practical way to calculate $\operatorname{rank} t$: Find a representation (2.3) of t which either contains no nonzero terms, in which case $\operatorname{rank} t = 0$, or which has the necessary linear independence properties, in which case $\operatorname{rank} t = n \geq 1$. Light and Cheney, in [LC85, p. 2], describe an iterative algorithm which finds such a representation as follows: Use one linear dependence at a time in conjunction with the bilinearity of $\otimes$ to reduce the number of terms in the representation (2.3) by one in each iteration; repeat this process until either no nonzero terms remain or the desired linear independence is achieved.

In Chapter 4, we will consider an alternative to Light and Cheney's iterative algorithm. This alternative uses a DIRECT[1] approach which finds the linear dependencies *all at once* and then eliminates them from the tensor product *all at once* rather than one at time in an iterative fashion. The author's DIRECT approach is better suited for efficient implementation in a computer algebra system—which, to be fair, is surely not what Light and Cheney had in mind!

All of the notations and definitions given earlier in this subsection are standard, and can be found scattered throughout the mathematical literature. For example, the tensor product of two *functions* is defined in [Edw65, p. 242, Dover], [Nie99, p. 39], and [CL00, p. 48]. The tensor product of two *function spaces* is defined in [HH91, pp. 266–267] and [CL00, p. 48].

The terminology, however, is not universal. Although tensor product representations of functions of two variables, such as representation (2.2), appear frequently as kernels in the theory of integral equations, they are not called tensor products in this context, but are known by a variety of other names: They are called **degenerate kernels** in [Sob64, p. 243, Dover], [Sta79, p. 352], [Arf85, pp. 882–883], [Che86, p. 18], [Kee88, p. 111], and [PS90, p. 57]; **separable kernels** in [Sta79, p. 352], [Arf85, pp. 882–883], [Che86, p. 18], and [Kee88, p. 111]; **kernels of finite rank** in [RSN55, p. 158, Dover] and [BF70, v. 2, pp. 504, 508, Dover]; **Pincherle-Goursat kernels** in [Tri57, pp. 55–56, Dover]; and **degenerated kernels** in [Yos60, p. 90, Dover].

What are we to make of this unruly state of affairs? The great Shakespeare once wrote, "What's in a name? That which we call a tensor product, by any other name, would

[1] DIRECT is an acronym. Your overwhelming curiosity about what it stands for *compels* you to keep on reading, doesn't it?!

facilitate the separation of variables with equal ease and thus still play an important role in a great variety of mathematical applications of enduring interest."[2] Ah, the classics!

[2]This quotation is from a very *early* draft of *Romeo and Juliet*. The mathematical material was deemed too scandalous and ultimately expurgated at the demand of the censors. Fortunately, the entire revision process has been painstakingly reconstructed in *The Collected Mathematical Writings of William Shakespeare: The Lost Years*. Unfortunately, this compelling work of postmodern scholarship is now permanently out of print, and all published copies have been destroyed by a government conspiracy—except for the few that escaped on the mother ship with the alien abductees—but you didn't hear that from me! Suffice it to say that the truth is out there somewhere.

Chapter 3

Foundations from Functional Analysis

In this chapter, we will build on the purely algebraic structures of the previous chapter by adding topological structures in order to develop some of the basic ideas of functional analysis. We shall still follow the rubric that says one should avoid invoking extraneous mathematical structure whenever possible; consequently, all subsequent notions which do not require topology will continue to be introduced in a purely algebraic fashion.

The first section of this chapter discusses the topological spaces of central importance to the thesis. The extended real numbers are introduced as the natural topological setting for asymptotic analysis in one real variable. Dual spaces play an important role in the thesis, and are developed here in a novel way; we will develop the fundamental properties of both *algebraic* and *topological* dual spaces, and then *unify* these properties to create the notion of a *generalized* dual space. We will use the generalized dual space throughout the thesis to develop dual-purpose results which are independent of any particular notion of the dual space. We will conclude the first section of this chapter by introducing all the normed linear spaces of real-valued functions which the thesis will require.

The second section of this chapter develops two purely algebraic formalisms with numerous applications to analysis: the cross-sections of bivariate functions and the parametric extensions of linear operators to spaces of bivariate functions. In the remainder of this chapter, we will use these formalisms to develop familiar examples of important linear operators, such as point-evaluation operators, and ordinary, partial, and normal derivative operators.

3.1 Topological Spaces

We will endow the real numbers $\mathbb{R}$ with the norm topology of the absolute value function, and we will denote open and closed intervals in $\mathbb{R}$ by (a, b) and $[a, b]$, respectively. Let $\overline{\mathbb{R}} := [-\infty, \infty]$ denote the set of extended real numbers with the topology of the two-point compactification. We may regard $\mathbb{R}$ as a topological subspace of $\overline{\mathbb{R}}$. Note that all topological operations on sets of real numbers are taken with respect to the topology of $\overline{\mathbb{R}}$ unless explicitly stated otherwise. For example, the closure of the open interval $(0, \infty)$ is by default $[0, \infty]$, not $[0, \infty)$.

Let $\mathbb{R}^d$ denote d-dimensional Euclidean space with the standard inner product. Let $\overline{\mathbb{R}}^d$ have the d-fold product topology of the topology on $\overline{\mathbb{R}}$. We may regard $\mathbb{R}^d$ as a topological subspace of $\overline{\mathbb{R}}^d$. If $S \subset \overline{\mathbb{R}}^d$, denote the closure of S by $\bar{S}$, the boundary of S by ∂S, and the interior of S by S°. Note that all of these topological operations are taken with respect to the topology of $\overline{\mathbb{R}}^d$. For example, $\overline{(0, \infty) \times (0, \infty)} = [0, \infty] \times [0, \infty]$, not $[0, \infty) \times [0, \infty)$.

For complete descriptions of the topological constructions introduced above, please consult the standard references [Kel55], [Mun75], [Roy68], and [Rud74]. In the next subsection, we will use both algebraic and topological constructions to develop three different notions of the dual of a vector space.

3.1.1 Algebraic, Topological, and Generalized Dual Spaces

There are two different standard notions of the dual space of a vector space—one is purely algebraic and the other depends on topology. In this subsection, we will develop both notions—and then *unify them by abstracting the shared properties which support our work with tensor products*. This will lead to a third, more general notion of the dual space which will include both of the first two notions as special cases. This unified, axiomatic approach draws upon many well-known mathematical techniques—but organizes and presents this standard material in a novel way that is original to the author.[1]

Let V be an arbitrary vector space over a field F. We define the **algebraic dual of** V, denoted by V^*, to be the set of all F-linear functionals on V:

$$V^* := \mathrm{Hom}_F(V, F).$$

The algebraic dual V^* is a function space over the field F. By analogy to classical inner

[1] A much cheekier author would have entitled this subsection, *All the Simple Things I Wish I Had Been Told about Dual Spaces (but Had to Figure Out for Myself by Reading Difficult Books).*

product spaces, it is customary to define a bilinear form

$$\langle \bullet, \bullet \rangle : V \times V^* \to F \quad \text{by} \quad \langle v, \phi \rangle := \phi(v)$$

for all $v \in V$ and all $\phi \in V^*$.

Now let V be an n-dimensional vector space over F, where $n \geq 1$, and let $B := \{v_i\}_{i=0}^{n-1}$ be a basis of V. We can define a subset $B^* := \{\phi_j\}_{j=0}^{n-1}$ of the algebraic dual V^* by prescribing the values of each ϕ_j on the basis B as follows:

$$\phi_j(v_i) := \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{for} \quad 0 \leq i, j \leq n-1.$$

The set B^* is a basis of V^*, called the **basis dual to** B. It follows that V and V^* have the same dimension over F:

$$\dim_F V := |B| = n = |B^*| =: \dim_F V^*.$$

The relationship between the basis B and the dual basis B^* can be expressed in terms of the bilinear form as

$$\langle v_i, \phi_j \rangle = \delta_{ij} \quad \text{for} \quad 0 \leq i, j \leq n-1.$$

For this reason, we say that the bases B and B^* are **biorthonormal**.[2] In the same way that a classical inner product computes the coefficients of a vector with respect to an orthonormal basis, the bilinear form $\langle \bullet, \bullet \rangle$ computes the coefficients of any vector $v \in V$ with respect to the basis B and the coefficients of any linear functional $\phi \in V^*$ with respect to the dual basis B^*. This property is expressed by the expansions

$$v = \sum_{i=0}^{n-1} \langle v, \phi_i \rangle \, v_i \quad \text{and} \quad \phi = \sum_{i=0}^{n-1} \langle v_i, \phi \rangle \, \phi_i.$$

We will now use the duality between V and V^* to define a fundamental property of *useful* subspaces of linear functionals $V^\dagger \subset V^*$ on an *arbitrary* (possibly infinite-dimensional) vector space V. Afterwards, we will show that *both* the algebraic *and* the topological notions of the dual space possess this important property.

[2]The author respectfully suggests that the stronger term "biorthonormal" [Dav63, p. 34, Dover] is more fitting in this context than the weaker term "biorthogonal" [CL00, p. 41]. In common mathematical usage, "orthonormal" means "orthogonal" and "normalized," which are two independent concepts; for example, an *orthogonal* basis in a classical inner product space need *not* be normalized.

Definition 3.1 *Let V be an arbitrary vector space over a field F, let V^* denote the algebraic dual of V, and let $V^\dagger \subset V^*$ denote any proper or improper subspace.*

1. *Given two sets $\{v_i\}_{i=0}^{n-1} \subset V$ and $\{\phi_j\}_{j=0}^{n-1} \subset V^*$, where $n \geq 1$, we say that $\{\phi_j\}_{j=0}^{n-1}$* ***is dual to*** *$\{v_i\}_{i=0}^{n-1}$* ***in*** *$V^\dagger$ if*

$$\langle v_i, \phi_j \rangle = \delta_{ij} \quad \textit{for} \quad 0 \leq i, j \leq n-1 \quad \textit{and} \quad \{\phi_j\}_{j=0}^{n-1} \subset V^\dagger.$$

2. *We say that $V^\dagger$* ***has the duality property*** *if for each positive integer n and each linearly independent set of n vectors $\{v_i\}_{i=0}^{n-1} \subset V$, there exists a set of n linear functionals $\{\phi_j\}_{j=0}^{n-1} \subset V^*$ which is dual to $\{v_i\}_{i=0}^{n-1}$ in $V^\dagger$.*

If V is n-dimensional and $\{v_i\}_{i=0}^{n-1}$ is a basis of V, the statement that $\{\phi_j\}_{j=0}^{n-1}$ is dual to $\{v_i\}_{i=0}^{n-1}$ in V^* simply means that $\{\phi_j\}_{j=0}^{n-1}$ is the basis of V^* dual to the basis $\{v_i\}_{i=0}^{n-1}$ of V; *however, if V is infinite-dimensional, the statement that $\{\phi_j\}_{j=0}^{n-1}$ is dual to $\{v_i\}_{i=0}^{n-1}$ in V^* asserts a much stronger property.* Why is this? Biorthonormality still implies that both $\{v_i\}_{i=0}^{n-1}$ and $\{\phi_j\}_{j=0}^{n-1}$ are linearly independent sets, the same as in the finite-dimensional case; however, the crucial difference is that *each linear functional ϕ_j is now defined on the whole infinite-dimensional vector space* V, not merely on the n-dimensional subspace $V_n := \operatorname{span}_F\{v_i\}_{i=0}^{n-1}$. The *restrictions* $\{\phi_j|V_n\}_{j=0}^{n-1}$, however, do yield the basis of V_n^* dual to the basis $\{v_i\}_{i=0}^{n-1}$ of V_n. The construction of dual bases by restriction will prove useful in Chapter 7 when we explore the precise nature of the duality in dual asymptotic expansions.

Informally speaking, the duality property tells us that the subspace $V^\dagger$ of the algebraic dual V^* is "sufficiently rich" to be of interest to us. Not every subspace has this property. For example, given a nontrivial vector space $V \neq \{0\}$, the trivial subspace $V^\dagger := \{0\}$ of the algebraic dual V^* does *not* have the duality property—thus, it is not "sufficiently rich." Can we always find a subspace of V^* which is "sufficiently rich"? We can indeed! The following proposition establishes that the algebraic dual V^* itself *always* has the duality property, even if V is infinite-dimensional:

Proposition 3.2 *If V is an arbitrary vector space over a field F, then the algebraic dual V^* has the duality property.*

Proof. If $V = \{0\}$, then $V^* = \{0\}$ has the duality property vacuously since V contains no linearly independent subsets. If $V \neq \{0\}$, then let n be any positive integer, and let $\{v_i\}_{i=0}^{n-1} \subset V$ be any linearly independent set of n vectors. Zorn's lemma[3] allows us

[3]Since Zorn's lemma is equivalent to the axiom of choice, this proof is nonconstructive.

to augment the set $\{v_i\}_{i=0}^{n-1}$ to form a Hamel basis $\{v_i\}_{i\in I}$ of V. (For the details of this construction, please consult [Lan65, pp. 85–86].) Although we assume for convenience that $\{0,1,\ldots,n-1\} \subset I$, we do *not* assume that $I \subset \mathbb{N}$, since I may be uncountable. Now define each $\phi_j \in V^*$ for $j = 0,1,\ldots,n-1$ by prescribing its values on the basis $\{v_i\}_{i\in I}$ in the following way:

$$\phi_j(v_i) := \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{for} \quad i \in I \quad \text{and} \quad 0 \leq j \leq n-1.$$

By construction, $\langle v_i, \phi_j\rangle = \delta_{ij}$ for $i,j = 0,1,\ldots,n-1$, and $\{\phi_j\}_{j=0}^{n-1} \subset V^*$, which means that $\{\phi_j\}_{j=0}^{n-1}$ is dual to $\{v_i\}_{i=0}^{n-1}$ in V^*, as required. ■

Now let $(V, \|\bullet\|)$ be a normed linear space over the field $\mathbb{R}$, and let V^* denote the algebraic dual of V, as above. The norm $\|\bullet\|$ on V induces a functional $\|\bullet\| : V^* \to [0,\infty]$ which is defined for every $\phi \in V^*$ by

$$\|\phi\| := \sup\{|\phi(v)| : v \in V \text{ with } \|v\| = 1\}.$$

This formula yields a well-defined functional on V^* as long as $V \neq \{0\}$. (If $V = \{0\}$, then $V^* = \{0\}$, in which case we define $\|\phi\| := 0$ for all $\phi \in V^*$.)

If $\phi \in V^*$ satisfies $\|\phi\| < \infty$, we say that ϕ **is bounded**; if $\|\phi\| = \infty$, we say that ϕ **is unbounded**. We define the **topological dual of** V, denoted by V', to be the set of all *bounded* linear functionals on V:

$$V' := \{\phi \in V^* : \|\phi\| < \infty\}.$$

This set is also known as the **continuous dual of** V. The terminology and notation used here for the algebraic dual space and the topological/continuous dual space can be found in [Edw65, pp. 42, 63, Dover] and [Pag78, p. 132, Dover].

The topological dual V' is a *subspace* of the algebraic dual V^*. The *restriction* of the induced functional $\|\bullet\|$ on V^* to the subspace V' is a norm on V', called the **operator norm**. The topological dual V' with the operator norm $\|\bullet\|$ is a *complete* normed linear space, or **Banach space**, over $\mathbb{R}$.

Let us now consider the nature of the relationship between the topological dual V' and the algebraic dual V^*. The following proposition establishes that V' actually *coincides* with V^* when V is *finite-dimensional:*

Proposition 3.3 *If $(V, \|\bullet\|)$ is a finite-dimensional normed linear space over the field $\mathbb{R}$, then the topological dual V' and the algebraic dual V^* are identical.*

Proof. We must prove that $V' = V^*$. By definition, $V' \subset V^*$. To prove that $V^* \subset V'$, we must show that every linear functional is bounded. Let $\phi \in V^*$ be an arbitrary linear functional. If $V = \{0\}$, then $V^* = \{0\}$, and $\|\phi\| := 0$; hence, ϕ is bounded. If $V \neq \{0\}$, define a real-valued functional on V by

$$\|v\|_\phi := \|v\| + |\phi(v)| \quad \text{for all} \quad v \in V,$$

and note that this functional is a norm on V. Since all norms on the finite-dimensional real vector space V are equivalent, there is a real constant $M > 0$ such that

$$\|v\|_\phi := \|v\| + |\phi(v)| \leq M \|v\| \quad \text{for all} \quad v \in V.$$

This implies that

$$|\phi(v)| \leq (M-1) \cdot \|v\| \quad \text{for all} \quad v \in V.$$

It follows that

$$\|\phi\| := \sup \{|\phi(v)| : v \in V \text{ with } \|v\| = 1\} \leq M - 1 < \infty,$$

and thus ϕ is bounded, as required. ∎

Inversely, the following proposition establishes that V' is *distinct* from V^* when V is *infinite-dimensional:*

Proposition 3.4 *If $(V, \|\bullet\|)$ is an infinite-dimensional normed linear space over the field $\mathbb{R}$, then the topological dual V' is a proper subspace of the algebraic dual V^*.*

Proof. We must construct an unbounded linear functional on V. Since V is infinite-dimensional, we know that $V \neq \{0\}$, and thus V has a Hamel basis. Let $\{v_i\}_{i \in I}$ be a Hamel basis of V, and assume without loss of generality that $\|v_i\| = 1$ for all $i \in I$. Since V is infinite-dimensional, the index set I contains a countably infinite subset; for convenience, assume that $\mathbb{N} \subset I$.

Define $\phi \in V^*$ by prescribing its values on the basis $\{v_i\}_{i \in I}$ as follows:

$$\phi(v_i) := \begin{cases} i & \text{if } i \in \mathbb{N} \\ 0 & \text{if } i \notin \mathbb{N} \end{cases} \quad \text{for all} \quad i \in I.$$

Since the Hamel basis $\{v_i\}_{i\in I}$ is normalized, and since $\mathbb{N} \subset I$, the set of all unit vectors contains the following countable subset:

$$\{v \in V : \|v\| = 1\} \supset \{v_i : i \in \mathbb{N}\}.$$

If we construct the image of these two sets of vectors under the map $v \mapsto |\phi(v)|$, we obtain the inclusion

$$\{|\phi(v)| : v \in V \text{ with } \|v\| = 1\} \supset \{|\phi(v_i)| : i \in \mathbb{N}\} = \{|i| : i \in \mathbb{N}\} = \mathbb{N}$$

by the definition of ϕ. If we now take the supremum of these two sets of real numbers, we obtain the inequality

$$\|\phi\| := \sup\{|\phi(v)| : v \in V \text{ with } \|v\| = 1\} \geq \sup \mathbb{N} = \infty.$$

This proves that $\|\phi\| = \infty$, as required. ■

The previous two propositions together yield the following corollary, which characterizes the property $V' = V^*$ by means of the simpler property $\dim_{\mathbb{R}} V < \infty$:

Corollary 3.5 *If $(V, \|\bullet\|)$ is a normed linear space over the field $\mathbb{R}$, then the topological dual V' and the algebraic dual V^* are identical if and only if V is finite-dimensional.*

Now that we fully understand the nature of the relationship between V' and V^*, we are ready to continue with our program of abstraction. The following proposition shows that regardless of whether V' and V^* are the same or different, *the topological dual V' always has the same duality property as the algebraic dual V^*:*

Proposition 3.6 *If $(V, \|\bullet\|)$ is a normed linear space over the field $\mathbb{R}$, then the topological dual V' has the duality property.*

Proof. If $V = \{0\}$, then $V' = V^* = \{0\}$ has the duality property vacuously since V contains no linearly independent subsets. If $V \neq \{0\}$, then let n be any positive integer, and let $\{v_i\}_{i=0}^{n-1} \subset V$ be any linearly independent set of n vectors. Define the n-dimensional subspace $V_n := \operatorname{span}_{\mathbb{R}}\{v_i\}_{i=0}^{n-1}$, and let $\{\psi_j\}_{j=0}^{n-1}$ be the basis of V_n^* dual to the basis $\{v_i\}_{i=0}^{n-1}$ of V_n. Since V_n is finite-dimensional, Proposition 3.3 implies that $V_n' = V_n^*$, and thus $\{\psi_j\}_{j=0}^{n-1} \subset V_n'$. The Hahn-Banach extension theorem[4] asserts that every bounded linear

[4]Since the Hahn-Banach extension theorem is a consequence of Zorn's lemma, the proof of this proposition is just as nonconstructive as the proof of the duality property for the algebraic dual of an arbitrary vector space. At least we have philosophical parity between both flavors of abstract linear algebra!

functional on the subspace V_n can be extended isometrically to a bounded linear functional on the whole normed linear space V. (For the details of this construction, please consult [Mad88, pp. 132–135] or [Con90, pp. 77–81].) This means that for each $j = 0, 1, \ldots, n-1$, there exists $\phi_j \in V'$ such that $\phi_j | V_n = \psi_j$ and $\left\|\phi_j\right\| = \left\|\psi_j\right\|$. By construction,

$$\langle v_i, \phi_j \rangle = \phi_j(v_i) = \psi_j(v_i) = \langle v_i, \psi_j \rangle = \delta_{ij} \quad \text{for} \quad i, j = 0, 1, \ldots, n-1,$$

and $\{\phi_j\}_{j=0}^{n-1} \subset V'$, which means that $\{\phi_j\}_{j=0}^{n-1}$ is dual to $\{v_i\}_{i=0}^{n-1}$ in V', as required. ■

We will now define *another* fundamental property of useful subspaces of linear functionals $V^\dagger \subset V^*$ on an arbitrary vector space V. Afterwards, we will show that *both* the algebraic dual V^* *and* the topological dual V' possess this important property as well.

Definition 3.7 *Let V be an arbitrary vector space over a field F, let V^* denote the algebraic dual of V, and let $V^\dagger \subset V^*$ denote any proper or improper subspace.*

1. *Given a vector $v \in V$, we say that $V^\dagger$ **annihilates** v if $\phi(v) = 0$ for all $\phi \in V^\dagger$, or equivalently, if*
$$v \in \bigcap_{\phi \in V^\dagger} \ker \phi.$$

2. *We say that $V^\dagger$ **has the annihilation property** if the only vector $v \in V$ annihilated by $V^\dagger$ is $v = 0$, or equivalently, if*
$$\bigcap_{\phi \in V^\dagger} \ker \phi = \{0\}.$$

The following proposition establishes that *the annihilation property is an immediate consequence of the duality property:*

Proposition 3.8 *Let V be an arbitrary vector space over a field F, let V^* denote the algebraic dual of V, and let $V^\dagger \subset V^*$ denote any proper or improper subspace. If $V^\dagger$ has the duality property, then $V^\dagger$ also has the annihilation property.*

Proof. If $V = \{0\}$, then $V^\dagger = V^* = \{0\}$ has the duality property vacuously and the annihilation property trivially. The proof in the case $V \neq \{0\}$ proceeds by contradiction: Let $V^\dagger$ annihilate the vector $v \in V$, and suppose that $v \neq 0$. In that case, $\{v\} \subset V$ is a linearly independent set. By the duality property of $V^\dagger$, there is a set $\{\phi\} \subset V^*$ such that $\{\phi\}$ is dual to $\{v\}$ in $V^\dagger$. This implies that $\phi(v) = 1$ and $\phi \in V^\dagger$; however, since $V^\dagger$

annihilates v, we must have $\phi(v) = 0$, which is a contradiction. Since supposing that $v \neq 0$ lead to a contradiction, it must be the case that $v = 0$, which implies that $V^\dagger$ has the annihilation property. ■

Corollary 3.9 *If V is an arbitrary vector space over a field F, then the algebraic dual V^* has the annihilation property.*

Corollary 3.10 *If $(V, \|\bullet\|)$ is a normed linear space over the field $\mathbb{R}$, then the topological dual V' has the annihilation property.*

In summary, we have established that the algebraic dual of an arbitrary vector space and the topological dual of a real normed linear space both have the duality property, which in turn implies the annihilation property. Since every subspace $V^\dagger \subset V^*$ which has the duality property *has two essential properties shared by the algebraic dual V^* and the topological dual V'*, we are motivated to make the following definition:

Definition 3.11 *Let V be an arbitrary vector space over a field F. We say that $V^\dagger$ is a* ***generalized dual of*** *V if*

1. *$V^\dagger$ is a proper or improper subspace of the algebraic dual V^*, and*

2. *$V^\dagger$ has the duality property.*

Clearly, the algebraic dual V^* and the topological dual V' are two concrete examples of a generalized dual $V^\dagger$. Working over a generalized dual space will allow us to develop tensor product theory *in a uniform fashion*, covering the algebraic and topological settings *both at once*. For example, in Chapter 4, we will use the annihilation property to prove a cancellation lemma for the tensor product of abstract vector spaces.

In addition, by formulating all of our work with tensor products in terms of these abstract properties, we will *automatically* obtain two interesting corollaries from *every* theorem involving a generalized dual $V^\dagger$: one corollary for the algebraic dual V^* and one corollary for the topological dual V'. *In effect, the axiomatic framework of this subsection automatically gives us three mathematical results for the price of one!*

3.1.2 Normed Linear Spaces of Real-Valued Functions

This subsection introduces the most important linear spaces of real-valued functions for the purposes of the thesis. Let X denote an arbitrary set. We can define a functional $\|\bullet\|_\infty : \mathbb{R}^X \to [0, \infty]$ by

$$\|f\|_\infty := \sup_{x \in X} |f(x)| \quad \text{for all} \quad f \in \mathbb{R}^X.$$

If $\|f\|_\infty < \infty$, we say that f **is bounded**. We denote the set of all real-valued bounded functions on X by $B(X)$. The set $B(X)$ is a subalgebra of the real function algebra $\mathbb{R}^X$. In addition, the functional $\|\bullet\|_\infty$ is a norm on the function space $B(X)$, called the **infinity norm**, the **norm of uniform convergence**, or simply the **uniform norm**; thus, $(B(X), \|\bullet\|_\infty)$ is a normed linear space over $\mathbb{R}$.

Now let $X \subset \overline{\mathbb{R}}^d$ for some integer $d \geq 1$. We will denote the set of all real-valued continuous functions on X by $C(X)$. The set $C(X)$ is a subalgebra of the real function algebra $\mathbb{R}^X$. If X is *compact*, then $C(X)$ is also a subalgebra of the function algebra $B(X)$, and the infinity norm restricted to $C(X)$ reduces to

$$\|f\|_\infty := \max_{x \in X} |f(x)| \quad \text{for all} \quad f \in C(X).$$

In addition, the normed linear space $(C(X), \|\bullet\|_\infty)$ is a Banach space over $\mathbb{R}$ whenever X is compact.

Let $G \subset \mathbb{R}^d$ be open and let $n \in \mathbb{N}$. We will denote the set of all real-valued functions on G with continuous partial derivatives up to order n at every point of G by $C^n(G)$. By convention, $C^0(G) := C(G)$. We will denote the set of all real-valued infinitely-differentiable functions on G by $C^\infty(G)$. We will denote the set of all real-valued real-analytic[5] functions on G by $C^\omega(G)$. These real function algebras are related to one another according to the following hierarchy:

$$C^\omega(G) \subset C^\infty(G) \subset C^n(G) \subset C(G) \subset \mathbb{R}^G.$$

Let $[a_i, b_i] \subset \mathbb{R}$ for $i = 0, 1, \ldots, d-1$, and define a compact d-dimensional hyperrectangle $K \subset \mathbb{R}^d$ via

$$K := [a_0, b_0] \times [a_1, b_1] \times \cdots \times [a_{d-1}, b_{d-1}].$$

[5]A real-valued function f is said to be real-analytic in d real variables on an open set $G \subset \mathbb{R}^d$ if f is represented by a convergent real power series in d real variables in a neighborhood of every point of G.

In this context, $C^n(K)$ denotes the set of all real-valued functions on K with continuous partial derivatives up to order n at every point of K, *but it is understood that we use one-sided partial derivatives as needed when constrained by the geometry of K at a point on the boundary ∂K.*

For example, in the case of the unit square $K := [0,1] \times [0,1]$, we define the partial derivatives of $f \in C^1(K)$ at the corner point $(0,0)$ using *one-sided limits in both x and y*, as follows:

$$\frac{\partial f}{\partial x}(0,0) := \lim_{x \to 0^+} \frac{f(x,0) - f(0,0)}{x} \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) := \lim_{y \to 0^+} \frac{f(0,y) - f(0,0)}{y}.$$

The effects of the boundary geometry on limits at the corner point $(0,0)$ of the unit square are illustrated in Figure 3.1.

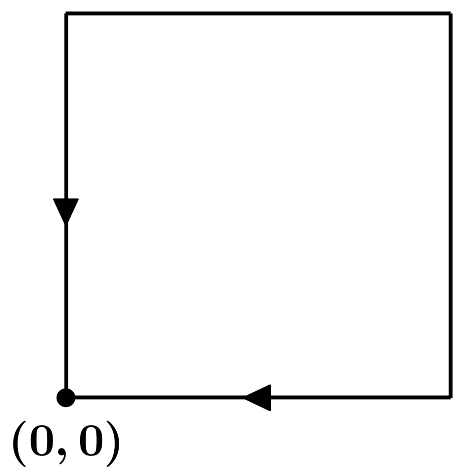

Figure 3.1: Limits at the Corner Point $(0,0)$ of the Unit Square $[0,1] \times [0,1]$

In contrast, at the point $(\frac{1}{2}, 0)$, which lies in the interior of the edge $[0,1] \times \{0\}$, we define the partial derivatives using a *two-sided limit in x* and a *one-sided limit in y*, as follows:

$$\frac{\partial f}{\partial x}\left(\frac{1}{2}, 0\right) := \lim_{x \to \frac{1}{2}} \frac{f(x,0) - f(0,0)}{x} \quad \text{and} \quad \frac{\partial f}{\partial y}\left(\frac{1}{2}, 0\right) := \lim_{y \to 0^+} \frac{f(\frac{1}{2},y) - f(\frac{1}{2},0)}{y}.$$

The effects of the boundary geometry on limits at the interior edge point $(\frac{1}{2}, 0)$ of the unit square are illustrated in Figure 3.2.

For a two-dimensional rectangle $K := [a_0, b_0] \times [a_1, b_1]$, an argument given in [Rud76,

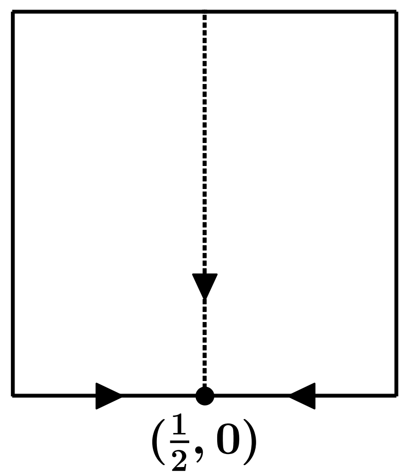

Figure 3.2: Limits at the Interior Edge Point $(\frac{1}{2}, 0)$ of the Unit Square $[0,1] \times [0,1]$

pp. 235–236] can be adapted to show that if $f \in C^2(K)$, then

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}$$

at every point of K, *including points on the boundary* ∂K. We can extend this line of reasoning to show that the mixed partial derivatives of order $i \in \{2, 3, \ldots, n\}$ for functions in $C^n(K)$ are equal *even on points of the boundary* ∂K in the d-dimensional case as well. Readers who wish to avoid such technical details may prefer to postulate the existence of an open set G such that $K \subset G \subset \mathbb{R}^d$, and construct $C^n(K)$ via restriction in the following way:

$$C^n(K) := \{f|K : f \in C^n(G)\}.$$

We conclude this subsection by noting that since $C^n(K)$ is a subspace of the normed linear space $(C(K), \|\bullet\|_\infty)$, we may regard $(C^n(K), \|\bullet\|_\infty)$ as a normed linear space as well—whenever $K \subset \mathbb{R}^d$ is a compact d-dimensional hyperrectangle.

3.2 Cross-Sections and Parametric Extensions

This section introduces a simple algebraic formalism which we will find enormously useful throughout the thesis. Let X and Y be arbitrary sets, and let F be an arbitrary field. Given a bivariate function $f : X \times Y \to F$, we can define two univariate functions $f_x : Y \to F$

and $f^y : X \to F$ by

$$f_x(y) := f(x,y) \quad \text{and} \quad f^y(x) := f(x,y) \quad \text{for all} \quad x \in X \quad \text{and} \quad y \in Y.$$

We call the univariate functions f_x and f^y the **x-section and y-section of** f, respectively, or simply the **cross-sections of** f. Note that for any fixed $x \in X$ and $y \in Y$, the maps $f \mapsto f_x$ and $f \mapsto f^y$ are *linear*; hence, we can write

$$(f \mapsto f_x) \in \mathrm{Hom}_F(F^{X\times Y}, F^Y) \quad \text{and} \quad (f \mapsto f^y) \in \mathrm{Hom}_F(F^{X\times Y}, F^X).$$

The notations f_x and f^y follow the conventions of real analysis exemplified in [Fri70, p. 84, Dover], [Rud74, p. 147], and [CL00, p. 54]. To avoid confusion with other common notations, please note that f_x does *not* denote the partial derivative of f with respect to x, and f^y does *not* denote f raised to the power y. These unsavory interpretations are hereby banished from the thesis!

For ease of reference, define the function spaces $U := F^X$, $V := F^Y$, and $W := F^{X\times Y}$. Given linear operators $S \in \mathrm{Hom}_F(U,U)$ and $T \in \mathrm{Hom}_F(V,V)$, the cross-section formalism allows us to define new linear operators $\tilde{S}, \tilde{T} \in \mathrm{Hom}_F(W,W)$ for any $f \in W$ by

$$(\tilde{S}f)(x,y) := (Sf^y)(x) \quad \text{and} \quad (\tilde{T}f)(x,y) := (Tf_x)(y) \quad \text{for all} \quad (x,y) \in X \times Y.$$

We call $\tilde{S}$ and $\tilde{T}$ the **parametric extensions of** S **and** T **to** W. Similar terminology can be found in [CL00, p. 54].

The process of parametric extension formalizes a common conceptual device: We often use operators on *univariate* functions (e.g., S and T) to define operators on *multivariate* functions (e.g., $\tilde{S}$ and $\tilde{T}$) by distinguishing one variable in particular and regarding the other variables as parameters held constant. For example, we defined $\tilde{S}$ in terms of S by treating x as the independent variable and regarding y as a parameter held constant; when we defined $\tilde{T}$ in terms of T, we simply interchanged the roles of x and y. The next subsection illustrates this formalism with a familiar example: ordinary and partial differentiation.

In what sense are $\tilde{S}$ and $\tilde{T}$ extensions? We can regard U and V as subspaces of W via the identifications $u(x,y) := u(x)$ and $v(x,y) := v(y)$ for all $u \in U$, $v \in V$, and $(x,y) \in X \times Y$. These identifications allow us to simplify the relevant cross-sections via $u^y = u$ and $v_x = v$. The operators $\tilde{S}$ and $\tilde{T}$ thus become true extensions of S and T in the sense that $\tilde{S}|U = S$ and $\tilde{T}|V = T$.

Similarly, we can regard F as a subspace of U and V via the identifications $\alpha(x) := \alpha$

and $\alpha(y) := \alpha$ for all $\alpha \in F$, $x \in X$, and $y \in Y$. This in turn allows us to regard the algebraic dual $U^* := \mathrm{Hom}_F(U, F)$ as a subspace of $\mathrm{Hom}_F(U, U)$, and the algebraic dual $V^* := \mathrm{Hom}_F(V, F)$ as a subspace of $\mathrm{Hom}_F(V, V)$. Given linear functionals $\phi \in U^*$ and $\psi \in V^*$, these identifications allow us to apply the general parametric extension process described above (with S, T, $\tilde{S}$, $\tilde{T}$ replaced by ϕ, ψ, Φ, Ψ); this yields linear operators $\Phi, \Psi \in \mathrm{Hom}_F(W, W)$ defined for any $f \in W$ by

$$(\Phi f)(x, y) := \phi(f^y) \quad \text{and} \quad (\Psi f)(x, y) := \psi(f_x) \quad \text{for all} \quad (x, y) \in X \times Y.$$

We will adopt the general convention of denoting linear functionals by *lowercase* Greek letters and their parametric extensions by the corresponding *uppercase* Greek letters. We will apply the parametric extension process for both linear operators and linear functionals in the next three subsections to develop a variety of linear operators that we will find useful throughout the thesis.

3.2.1 Ordinary and Partial Derivative Operators

Given intervals $X, Y \subset \mathbb{R}$, we will denote the *ordinary derivatives* of univariate functions in $C^1(X)$ and $C^1(Y)$ using the operators

$$D_x := \frac{d}{dx} \quad \text{and} \quad D_y := \frac{d}{dy}$$

respectively. In contexts where there is no need to distinguish between different independent variables, we will use the general univariate derivative operator D. We will denote the *partial derivatives* of bivariate functions in $C^1(X \times Y)$ using the operators

$$\partial_x := \frac{\partial}{\partial x} \quad \text{and} \quad \partial_y := \frac{\partial}{\partial y}.$$

Note that the first-order partial derivatives of a function $f \in C^1(X \times Y)$ can be expressed in the cross-section notation as

$$(\partial_x f)(x, y) = (D_x f^y)(x) \quad \text{and} \quad (\partial_y f)(x, y) = (D_y f_x)(y).$$

This shows that the partial derivative operators ∂_x and ∂_y are parametric extensions of the ordinary derivative operators D_x and D_y.

3.2.2 Evaluation Functionals and Evaluation Operators

Let X and Y be arbitrary sets, and let F be an arbitrary field. Define the function spaces $U := F^X$, $V := F^Y$, and $W := F^{X\times Y}$. Given parameters $x_0 \in X$ and $y_0 \in Y$, we define the **evaluation functionals at x_0 and y_0**, denoted by ε_{x_0} and ε^{y_0} respectively, via

$$\varepsilon_{x_0}(u) := u(x_0) \quad \text{and} \quad \varepsilon^{y_0}(v) := v(y_0) \quad \text{for all} \quad u \in U \quad \text{and} \quad v \in V.$$

Note that ε_{x_0} and ε^{y_0} are *linear* functionals; hence, $\varepsilon_{x_0} \in U^*$ and $\varepsilon^{y_0} \in V^*$.

The linear functionals ε_{x_0} and ε^{y_0} perform evaluation on *univariate* functions. In order to perform evaluation on *bivariate* functions—one variable at a time—we must define the parametric extensions of ε_{x_0} and ε^{y_0} to W. We denote these parametric extensions by E_{x_0} and E^{y_0}, respectively, and define these operators for any $f \in W$ via

$$(E_{x_0}f)(x,y) := f(x_0,y) \quad \text{and} \quad (E^{y_0}f)(x,y) := f(x,y_0) \quad \text{for all} \quad (x,y) \in X \times Y.$$

Since the parametric extensions of *linear* functionals are always *linear* operators, we can write E_{x_0}, $E^{y_0} \in \mathrm{Hom}_F(W,W)$. We call E_{x_0} and E^{y_0} the **evaluation operators at x_0 and y_0**.

We can reveal a connection between the evaluation operators and the cross-section formalism by eliminating the independent variables in the equations that define E_{x_0} and E^{y_0} and retaining only the parameters. This yields

$$E_{x_0}f = f_{x_0} \quad \text{and} \quad E^{y_0}f = f^{y_0} \quad \text{for all} \quad f \in W.$$

We now recognize the evaluation operators E_{x_0} and E^{y_0} as the familiar linear maps

$$(f \mapsto f_{x_0}) \in \mathrm{Hom}_F(W,V) \quad \text{and} \quad (f \mapsto f^{y_0}) \in \mathrm{Hom}_F(W,U)$$

respectively. This reformulation makes it clear that the evaluation operators E_{x_0} and E^{y_0} transform *bivariate* functions in W into *univariate* functions in V and U respectively.

By regarding V and U as subspaces of W, we recover the original definition of E_{x_0} and E^{y_0} as elements of $\mathrm{Hom}_F(W,W)$. This point of view allows us to *compose* the operators E_{x_0} and E^{y_0}. Upon doing so, we discover that these two operators *commute*; this property follows from the operator relations $E_{x_0}f = f_{x_0}$ and $E^{y_0}f = f^{y_0}$, the restriction properties

$E_{x_0}|U = \varepsilon_{x_0}$ and $E^{y_0}|V = \varepsilon^{y_0}$, and our prior definitions, which together yield

$$(E_{x_0}E^{y_0})(f) = \varepsilon_{x_0}(f^{y_0}) := f^{y_0}(x_0) := f(x_0, y_0) =: f_{x_0}(y_0) =: \varepsilon^{y_0}(f_{x_0}) = (E^{y_0}E_{x_0})(f)$$

for all $f \in W$. Working at this level of abstraction, we see immediately that

$$(E_{x_0}E^{y_0})(f) = f(x_0, y_0) \quad \text{for all} \quad f \in W.$$

Since the composition $E_{x_0}E^{y_0}$ takes on values in the field F, this composition is actually a linear *functional* on the function space W; hence, $E_{x_0}E^{y_0} \in W^*$.

In summary, we started with linear functionals ε_{x_0} and ε^{y_0} on the *univariate* function spaces U and V, applied the parametric extension process to obtain *commuting* linear operators E_{x_0} and E^{y_0} on the *bivariate* function space W, and then *composed* these operators to obtain a *new linear functional* $E_{x_0}E^{y_0}$ on W. We will repeat this general pattern many times throughout the thesis.

3.2.3 Normal Derivative Operators

Let $X, Y \subset \mathbb{R}$ be intervals, and select a point $(x_0, y_0) \in X \times Y$. Let $u \in C^1(X)$, $v \in C^1(Y)$, and $f \in C^1(X \times Y)$ be arbitrary. The parametric extensions of the linear functionals

$$(\varepsilon_{x_0}D_x)(u) := \frac{du}{dx}(x_0) \quad \text{and} \quad (\varepsilon^{y_0}D_y)(v) := \frac{dv}{dy}(y_0)$$

are given by

$$(E_{x_0}\partial_x f)(y) := \frac{\partial f}{\partial x}(x_0, y) \quad \text{and} \quad (E^{y_0}\partial_y f)(x) := \frac{\partial f}{\partial y}(x, y_0)$$

for all $x \in X$ and $y \in Y$. We call the univariate functions $E_{x_0}\partial_x f$ and $E^{y_0}\partial_y f$ the **normal derivatives of f on the lines $x = x_0$ and $y = y_0$**, respectively, and $E_{x_0}\partial_x$ and $E^{y_0}\partial_y$ are the corresponding **normal derivative operators**.

We can define higher-order normal derivatives as follows: Let $n \in \mathbb{N}$, $u \in C^n(X)$, $v \in C^n(Y)$, and $f \in C^n(X \times Y)$ be arbitrary. The parametric extensions of the linear functionals

$$(\varepsilon_{x_0}D_x^n)(u) := \frac{d^n u}{dx^n}(x_0) \quad \text{and} \quad (\varepsilon^{y_0}D_y^n)(v) := \frac{d^n v}{dy^n}(y_0)$$

are given by

$$(E_{x_0}\partial_x^n f)(y) := \frac{\partial^n f}{\partial x^n}(x_0, y) \quad \text{and} \quad (E^{y_0}\partial_y^n f)(x) := \frac{\partial^n f}{\partial y^n}(x, y_0)$$

for all $x \in X$ and $y \in Y$. We call the univariate functions $E_{x_0}\partial_x^n f$ and $E^{y_0}\partial_y^n f$ the **n-th order normal derivatives of f on the lines $x = x_0$ and $y = y_0$**, respectively, and $E_{x_0}\partial_x^n$ and $E^{y_0}\partial_y^n$ are the corresponding **n-th order normal derivative operators**.

If $f \in C^{2n}(X \times Y)$, Clairaut's theorem states that

$$(E_{x_0}\partial_x^n)(E^{y_0}\partial_y^n)f = \frac{\partial^{2n} f}{\partial x^n \partial y^n}(x_0, y_0) = \frac{\partial^{2n} f}{\partial y^n \partial x^n}(x_0, y_0) = (E^{y_0}\partial_y^n)(E_{x_0}\partial_x^n)f.$$

In other words, the n-th order normal derivative operators $E_{x_0}\partial_x^n$ and $E^{y_0}\partial_y^n$ *commute.* Furthermore, the composition $(E_{x_0}\partial_x^n)(E^{y_0}\partial_y^n)$ is a *linear functional* on the function space $C^{2n}(X \times Y)$. This generalizes our earlier observations that the evaluation operators E_{x_0} and E^{y_0} commute and that the composition $E_{x_0}E^{y_0}$ is a linear functional.

Commuting pairs of parametric extensions play a major role in the author's program of research—particularly in the development of the abstract splitting operator. In [CL00, pp. 54–55], Cheney and Light prove the following general result under the assumption that X and Y are compact Hausdorff spaces: *Bounded* linear operators on the Banach spaces $(C(X), \|\bullet\|_\infty)$ and $(C(Y), \|\bullet\|_\infty)$ have parametric extensions to $(C(X \times Y), \|\bullet\|_\infty)$ which *always* commute; the proof uses the Riesz representation theorem and the Fubini integration theorem, both for signed measures. Note that for the special case of compact intervals $X, Y \subset \mathbb{R}$, the commuting of the partial derivative operators ∂_x and ∂_y on $C^2(X \times Y)$ does *not* follow from this general theorem; the reason is that the ordinary derivative operators D_x and D_y are *unbounded* operators with respect to the infinity norm on the subspaces $C^1(X) \subset C(X)$ and $C^1(Y) \subset C(Y)$. For the same reason, the commuting of the n-th order normal derivative operators $E_{x_0}\partial_x^n$ and $E^{y_0}\partial_y^n$ on $C^{2n}(X \times Y)$ does *not* follow from Cheney and Light's general result, and must be proven by other means.

The general process of parametric extension, along with the familiar examples of parametric extensions which we developed in this section, will serve us tremendously well in the remainder of the thesis. Indeed, in his own research on the interpolation of functions of two variables, the author has found the cross-section formalism and the process of parametric extension to be *the top two most useful elementary ideas of all time!*

Chapter 4

Tensor Product Normal Forms and Invariants

In Chapter 2, we learned the basic facts about algebraic tensor products of function spaces. In this chapter, we will revisit these basic facts in the more general setting that results when we replace concrete function spaces with abstract vector spaces. It is actually *easier* to develop algebraic tensor product theory at this higher level of abstraction thanks to the celebrated "universal property" of tensor products—a very simple but extremely powerful property which reduces the study of multilinear algebra to the study of linear algebra on tensor product spaces. Of course, concrete results for function spaces will follow immediately as special cases of the abstract results that we develop for arbitrary vector spaces.

We are particularly interested in developing a deeper understanding of the nonuniqueness of the expressions which represent tensor products. The central goal of this chapter is to develop techniques for managing this nonuniqueness effectively. Although there is probably no such thing as a canonical representation for tensor products, we can develop the next best thing: an effective way to determine whether any two given expressions represent the same tensor product. In this chapter, we will develop an entire suite of tools, called "normal forms" and "invariants," which will allow us to do this in a variety of ways—and all of these ways are totally algorithmic!

4.1 Tensor Products of Abstract Vector Spaces

In this section, we will introduce the algebraic tensor product of two abstract vector spaces and then discuss its basic properties. Our approach is similar to the approach used by Pierce in [Pie82, pp. 157–162]. The major difference is that Pierce develops algebraic tensor products in the more general setting of modules over a commutative ring. We, however, will not need the generality of modules—vector spaces will suffice for the purposes of this thesis. Let us begin with this fundamental definition:

Definition 4.1 *Let U and V be vector spaces over a field F. A* ***tensor product of*** U ***and*** V *is a vector space $U \otimes V$ over the field F along with a bilinear map*

$$U \times V \to U \otimes V \quad \text{denoted by} \quad (u, v) \mapsto u \otimes v$$

which together satisfy the following two properties:

1. *The vector space $U \otimes V$ is generated in the following way:*

$$U \otimes V := \operatorname{span}_F G \quad \text{where} \quad G := \{u \otimes v : u \in U,\ v \in V\}.$$

2. *Given any vector space W over the field F and any bilinear map*

$$B : U \times V \to W,$$

there exists a linear map

$$L : U \otimes V \to W$$

such that

$$L(u \otimes v) = B(u, v) \quad \text{for all} \quad u \in U \quad \text{and} \quad v \in V.$$

This is referred to as the ***universal property****, or simply as* ***universality****.*

The universal property of tensor products merely guarantees the *existence* of the linear map L; however, it quickly follows that the map L is also *unique*: If $L_1(u \otimes v) = B(u, v) = L_2(u \otimes v)$ for all $u \in U$ and $v \in V$, then the linear map $L_1 - L_2$ vanishes on the set of generators G, which means that $L_1 - L_2$ also vanishes on $U \otimes V$; hence, $L_1 = L_2$.

Remark 4.2 *From now on, we will consider the uniqueness of the linear map L to be an integral part of the universal property of $U \otimes V$.*

We can use the universal property in this extended sense to construct an isomorphism between any two tensor products $U \otimes V$ and $U \otimes' V$ of two given vector spaces U and V. In this context, we define a tensor product isomorphism as follows:

Definition 4.3 *Let U and V be vector spaces over a field F, and let $U \otimes V$ and $U \otimes' V$ be any two tensor products of U and V. A* ***tensor product isomorphism of $U \otimes V$ and $U \otimes' V$*** *is a vector space isomorphism*

$$L : U \otimes V \to U \otimes' V$$

such that

$$L(u \otimes v) = u \otimes' v \quad \textit{for all} \quad u \in U \quad \textit{and} \quad v \in V.$$

The following proposition shows that such an isomorphism always exists—and is in fact unique!

Proposition 4.4 *Let U and V be vector spaces over a field F, and let $U \otimes V$ and $U \otimes' V$ be any two tensor products of U and V. There exists a unique tensor product isomorphism of $U \otimes V$ and $U \otimes' V$.*

Proof. The following proof is a more detailed version of the proof given in [Pie82, p. 158]. By definition, the map $U \times V \to U \otimes' V$ given by $(u, v) \mapsto u \otimes' v$ is bilinear. By the universal property of $U \otimes V$, there exists a unique linear map $L : U \otimes V \to U \otimes' V$ such that $L(u \otimes v) = u \otimes' v$ for all $u \in U$ and $v \in V$. This proves *uniqueness*; in order to prove *existence*, it suffices to show that L is a bijection. By the bilinearity of $\otimes$ and the universality of $\otimes'$, there exists a unique linear map $\tilde{L} : U \otimes' V \to U \otimes V$ such that $\tilde{L}(u \otimes' v) = u \otimes v$ for all $u \in U$ and $v \in V$. Since

$$\tilde{L}L(u \otimes v) = \tilde{L}(u \otimes' v) = u \otimes v \quad \text{for all} \quad u \in U \quad \text{and} \quad v \in V,$$

the composition $\tilde{L}L$ is the identity map on the set of generators G, and thus on the entire space $U \otimes V$. Similarly, the composition $L\tilde{L}$ is the identity map on the entire space $U \otimes' V$. It follows that L is a bijection with $L^{-1} = \tilde{L}$. We conclude that L is the unique tensor product isomorphism of $U \otimes V$ and $U \otimes' V$. ∎

Since all tensor products of U and V are *unique* up to isomorphism, we can speak of "the" tensor product of U and V. A more involved but nevertheless standard construction also shows that the tensor product of U and V always *exists* for every pair of vector spaces U and V.

In addition to establishing the uniqueness of the tensor product of two vector spaces, the universal property has another magnificent virtue whose value cannot be overstated: The universal property allows us to define any *linear* map $L : U \otimes V \to W$ simply by specifying a *bilinear* map $B : U \times V \to W$ which calculates the values of L on the set of generators G. Given such a bilinear map, the existence of a unique linear map $L : U \otimes V \to W$ such that $L(u \otimes v) = B(u, v)$ for all $u \in U$ and $v \in V$ follows immediately by the universal property.

Remark 4.5 *Note in particular that a linear map which is defined by the universal property is automatically well-defined: The value of the linear map is guaranteed to be the same for all distinct—but equivalent—representations of its tensor product argument! The best part of all is that we do not need to do any extra work whatsoever to show that a linear map is well-defined when we use this technique; thus, the universal property of tensor products is truly a labor-saving device! Please, dear reader, do the whole world an enormous favor by using the universal property whenever humanly possible!!*[1]

The following proposition illustrates the extraordinary usefulness of the universal property for defining linear maps on tensor product spaces:

Proposition 4.6 *For $i = 1, 2$, let U_i and V_i be vector spaces over a field F. Assume that we are given the following linear maps:*

$$S : U_1 \to U_2 \quad \textit{and} \quad T : V_1 \to V_2.$$

There exists a unique linear map

$$S \otimes T : U_1 \otimes V_1 \to U_2 \otimes V_2$$

such that

$$(S \otimes T)(u \otimes v) = (Su) \otimes (Tv) \quad \textit{for all} \quad u \in U_1 \quad \textit{and} \quad v \in V_1.$$

Proof. Since $(u, v) \mapsto (Su) \otimes (Tv)$ is a bilinear map of $U_1 \times V_1 \to U_2 \otimes V_2$, the result follows immediately by the universal property of $U_1 \otimes V_1$. ■

[1] The author will now step down from his soapbox.

Consider the unique linear map $F \otimes V \to V$ such that $\alpha \otimes v \mapsto \alpha \cdot v$, which is defined by the universal property of $F \otimes V$ and the bilinearity of the map $F \times V \to V$ given by $(\alpha, v) \mapsto \alpha \cdot v$. This linear map is actually a vector space isomorphism of $F \otimes V$ and V, and its inverse is the linear map $V \to F \otimes V$ defined by $v \mapsto 1 \otimes v$.

Remark 4.7 *Since $F \otimes V$ and V are isomorphic vector spaces, we can identify their elements, and often do so by writing $\alpha \otimes v = \alpha \cdot v$. Since a similar argument shows that $V \otimes F$ and V are isomorphic vector spaces, we can identify their elements as well, and often do so by writing $v \otimes \alpha = \alpha \cdot v$.*

With these identifications, the previous proposition yields the following useful corollary:

Corollary 4.8 *Let U and V be vector spaces over a field F, and let I denote the identity maps on both U and V. (The correct interpretation of I will be clear from the context.) Given any linear functionals $\phi \in U^*$ and $\psi \in V^*$, there exist unique linear maps*

$$\phi \otimes I : U \otimes V \to V \quad \textit{and} \quad I \otimes \psi : U \otimes V \to U$$

such that

$$(\phi \otimes I)(u \otimes v) = \phi(u) \cdot v \quad \textit{and} \quad (I \otimes \psi)(u \otimes v) = \psi(v) \cdot u$$

for all $u \in U$ and $v \in V$.

Just as any two tensor products $U \otimes V$ and $U \otimes' V$ of two given vector spaces are isomorphic, it happens that the tensor products $U \otimes V$ and $V \otimes U$ of the same two vector spaces *in opposite orders* are also isomorphic. This relationship is fundamentally important because it will provide the basis for exploiting duality in the remainder of this chapter. The following proposition describes this important relationship in detail:

Proposition 4.9 *Given any vector spaces U and V over a field F, there exist unique vector space isomorphisms*

$$T : U \otimes V \to V \otimes U \quad \textit{and} \quad \tilde{T} : V \otimes U \to U \otimes V$$

such that

$$T(u \otimes v) = v \otimes u \quad \textit{and} \quad \tilde{T}(v \otimes u) = u \otimes v \quad \textit{for all} \quad u \in U \quad \textit{and} \quad v \in V.$$

Proof. This proof is very similar to the proof of the unique tensor product isomorphism between $U \otimes V$ and $U \otimes' V$, except that we will now compare $U \otimes V$ and $V \otimes U$. By the universality of $U \otimes V$ and the bilinearity of the map $U \times V \to V \otimes U$ given by $(u, v) \mapsto v \otimes u$, there exists a unique linear map

$$T : U \otimes V \to V \otimes U \quad \text{such that} \quad T(u \otimes v) = v \otimes u \quad \text{for all} \quad u \in U \quad \text{and} \quad v \in V.$$

Similarly, by the universality of $V \otimes U$ and the bilinearity of the map $V \times U \to U \otimes V$ given by $(v, u) \mapsto u \otimes v$, there exists a unique linear map

$$\tilde{T} : V \otimes U \to U \otimes V \quad \text{such that} \quad \tilde{T}(v \otimes u) = u \otimes v \quad \text{for all} \quad u \in U \quad \text{and} \quad v \in V.$$

This proves *uniqueness*; in order to prove *existence*, it suffices to show that T and $\tilde{T}$ are bijections. Since

$$\tilde{T}T(u \otimes v) = \tilde{T}(v \otimes u) = u \otimes v \quad \text{for all} \quad u \in U \quad \text{and} \quad v \in V,$$

the composition $\tilde{T}T$ is the identity map on the set of generators G, and thus on the entire space $U \otimes V$. Similarly, the composition $T\tilde{T}$ is the identity map on the entire space $V \otimes U$. It follows that the maps T and $\tilde{T}$ are bijections with $T^{-1} = \tilde{T}$. We conclude that T and $\tilde{T}$ are the unique vector space isomorphisms with the specified properties. ∎

The maps T and $\tilde{T}$ will play a role in this chapter sufficiently important to warrant the following definition:

Definition 4.10 *Let T and $\tilde{T}$ denote the vector space isomorphisms of the previous proposition. We will call the map T the* ***transpose operator on*** *$U \otimes V$, and we will call the inverse map $\tilde{T}$ the* ***transpose operator on*** *$V \otimes U$.*

Since $T(u \otimes v) = v \otimes u$, we see that the map T does indeed *transpose* the factors of any dyad. Furthermore, since $T(U \otimes V) = V \otimes U$, we see that the map T also *transposes* the factors of the tensor product space. The inverse map $\tilde{T}$ likewise *transposes* the factors of any dyad $v \otimes u$ and *transposes* the factors of the tensor product space $V \otimes U$ as well. These observations motivate the following definition:

Definition 4.11 *Each of the two dyads $u \otimes v$ and $v \otimes u$ is called the* ***transpose*** *of the other, and each of the two tensor product spaces $U \otimes V$ and $V \otimes U$ is called the* ***transpose*** *of the other.*

We will use the transpose operators T and $\tilde{T}$ primarily to exploit the *duality* that often exists between the *roles* of the factors U and V in a proposition concerning the tensor product space $U \otimes V$. We can often use this duality to give concise, rigorous proofs of many propositions which have two similar cases by reducing the second case to the first case via the transpose operators. We will now illustrate this technique by proving the much-anticipated *cancellation lemma for the tensor product of abstract vector spaces:*

Lemma 4.12 (Cancellation) *Let U and V be vector spaces over a field F, and let $\{u_i\}_{i=0}^{n-1} \subset U$ and $\{v_i\}_{i=0}^{n-1} \subset V$, where $n \geq 1$. Assume that*

$$\sum_{i=0}^{n-1} u_i \otimes v_i = 0. \tag{4.1}$$

The following two results hold independently:

1. *If $\{u_i\}_{i=0}^{n-1}$ is a linearly independent set, then all the elements of $\{v_i\}_{i=0}^{n-1}$ are zero.*
2. *If $\{v_i\}_{i=0}^{n-1}$ is a linearly independent set, then all the elements of $\{u_i\}_{i=0}^{n-1}$ are zero.*

Proof. In order to make this proof independent of any particular notion of the dual space, we will let $V^\dagger$ denote a *generalized dual space* of V. Assume that $\{u_i\}_{i=0}^{n-1}$ is a linearly independent set, and let $\psi \in V^\dagger$ be arbitrary. We previously constructed a linear map $I \otimes \psi : U \otimes V \to U$ such that $(I \otimes \psi)(u \otimes v) = \psi(v) \cdot u$ for all $u \in U$ and $v \in V$. Applying the linear map $I \otimes \psi$ to equation (4.1) yields

$$(I \otimes \psi)\left(\sum_{i=0}^{n-1} u_i \otimes v_i\right) = \sum_{i=0}^{n-1}(I \otimes \psi)\,(u_i \otimes v_i) = \sum_{i=0}^{n-1} \psi(v_i) \cdot u_i = 0.$$

Since $\{u_i\}_{i=0}^{n-1}$ is a linearly independent set, the coefficients in the previous equation must satisfy

$$\psi(v_i) = 0 \quad \text{for} \quad 0 \leq i \leq n-1.$$

Since $\psi \in V^\dagger$ was arbitrary, we can assert the following for $0 \leq i \leq n-1$:

$$\psi(v_i) = 0 \quad \text{for all} \quad \psi \in V^\dagger.$$

This means that $V^\dagger$ annihilates v_i for $0 \leq i \leq n-1$. By the annihilation property of the generalized dual space $V^\dagger$, we conclude that $v_i = 0$ for $0 \leq i \leq n-1$, as desired.

Now assume that $\{v_i\}_{i=0}^{n-1}$ is a linearly independent set. We previously constructed the transpose operator T on the tensor product space $U \otimes V$. Applying the linear map T to equation (4.1) yields

$$T\left(\sum_{i=0}^{n-1} u_i \otimes v_i\right) = \sum_{i=0}^{n-1} T\left(u_i \otimes v_i\right) = \sum_{i=0}^{n-1} v_i \otimes u_i = 0.$$

Since the linearly independent set $\{v_i\}_{i=0}^{n-1}$ is contained in the *first* factor of the *transposed* space $V \otimes U$, and since the set $\{u_i\}_{i=0}^{n-1}$ is contained in the *second* factor of $V \otimes U$, we can now apply the result of the previous paragraph! We conclude that all the elements of the set $\{u_i\}_{i=0}^{n-1}$ must be zero, as desired. ■

After successfully exploiting the duality between the tensor product space $U \otimes V$ and its transpose $V \otimes U$ to produce a concise, complete, and totally rigorous proof of the previous proposition, one thing should be clear:

Remark 4.13 *Duality is a truly labor-saving device! Please, dear reader, do the whole world a gigantic favor by using duality whenever humanly possible!!*[2]

The following familiar result is a very important consequence of the cancellation lemma:

Theorem 4.14 (Hamel Basis) *Let U and V be vector spaces over a field F. If $\{u_i\}_{i\in I}$ and $\{v_j\}_{j\in J}$ are Hamel bases of U and V, respectively, then $\{u_i \otimes v_j\}_{(i,j)\in I\times J}$ is a Hamel basis of $U \otimes V$.*

Proof. Let $B := \{u_i \otimes v_j\}_{(i,j)\in I\times J}$. We will first show that B generates $U \otimes V$. Since $U \otimes V$ is generated by $G := \{u \otimes v : u \in U,\ v \in V\}$ by definition, it suffices to show that every dyad $u \otimes v$ is a finite linear combination of elements of B. For every $u \in U$ and $v \in V$, there exist nonempty finite index sets $I_0 \subset I$ and $J_0 \subset J$ and sets of coefficients $\{\alpha_i\}_{i\in I_0}, \{\beta_j\}_{j\in J_0} \subset F$ such that

$$u = \sum_{i\in I_0} \alpha_i \cdot u_i \quad \text{and} \quad v = \sum_{j\in J_0} \beta_j \cdot v_j.$$

The bilinearity of $\otimes$ implies that

$$u \otimes v = \left(\sum_{i\in I_0} \alpha_i \cdot u_i\right) \otimes \left(\sum_{j\in J_0} \beta_j \cdot v_j\right) = \sum_{i\in I_0} \sum_{j\in J_0} \alpha_i \beta_j \cdot (u_i \otimes v_j).$$

[2] The author will *now* step down from his soapbox—really!

This shows that B generates G, which in turn generates $U \otimes V$.

We must now show that the elements of B are linearly independent, which means showing that the set $\{u_i \otimes v_j\}_{(i,j)\in S}$ is linearly independent for every nonempty finite index set $S \subset I \times J$. Choose an arbitrary nonempty finite index set $S \subset I \times J$, and let the coefficients $\{\gamma_{ij}\}_{(i,j)\in S} \subset F$ also be arbitrary. Assume that the following linear combination of the elements of $\{u_i \otimes v_j\}_{(i,j)\in S}$ is equal to zero:

$$\sum_{(i,j)\in S} \gamma_{ij} \cdot (u_i \otimes v_j) = 0. \tag{4.2}$$

In order to separate the index variables i and j in the previous equation, we will now embed the index set S in a suitable Cartesian product $I_0 \times J_0$ by the following construction: Let I_0 denote the set of all *first* coordinates of the elements of S, and let J_0 denote the set of all *second* coordinates of the elements of S. The sets $I_0 \subset I$ and $J_0 \subset J$ are nonempty and finite since S is nonempty and finite, and $S \subset I_0 \times J_0$ by construction. We can now *extend* the definition of the original coefficients $\{\gamma_{ij}\}_{(i,j)\in S} \subset F$ by letting

$$\gamma_{ij} := 0 \quad \text{for all} \quad (i,j) \in (I_0 \times J_0)\backslash S.$$

This allows us to replace the original index set S by the enlarged index set $I_0 \times J_0$ in order to rewrite equation (4.2) in a form which facilitates the separation of the index variables i and j:

$$\sum_{(i,j)\in S} \gamma_{ij} \cdot (u_i \otimes v_j) = \sum_{(i,j)\in I_0\times J_0} \gamma_{ij} \cdot (u_i \otimes v_j) = \sum_{i\in I_0}\sum_{j\in J_0} \gamma_{ij} \cdot (u_i \otimes v_j) = 0.$$

Using the bilinearity of $\otimes$, we can rewrite the previous equation as

$$\sum_{i\in I_0} u_i \otimes \left(\sum_{j\in J_0} \gamma_{ij} \cdot v_j \right) = 0.$$

Since the nonempty finite set $\{u_i\}_{i\in I_0}$ is linearly independent, Cancellation Lemma 4.12 implies that

$$\sum_{j\in J_0} \gamma_{ij} \cdot v_j = 0 \quad \text{for all} \quad i \in I_0.$$

The linear independence of the nonempty finite set $\{v_j\}_{j\in J_0}$ further implies that

$$\gamma_{ij} = 0 \quad \text{for all} \quad j \in J_0 \quad \text{and} \quad i \in I_0.$$

In particular, $\gamma_{ij} = 0$ for all $(i, j) \in S$, which shows that the set $\{u_i \otimes v_j\}_{(i,j)\in S}$ is linearly independent, as desired. ■

The previous theorem leads immediately to the following familiar corollary:

Corollary 4.15 *Given any vector spaces U and V over a field F, it follows that*

$$\dim_F(U \otimes V) = \dim_F U \cdot \dim_F V.$$

Remember that the dimension of an arbitrary vector space is a cardinal number, which means that we multiply vector space dimensions using cardinal arithmetic. Of course, if both U and V are finite-dimensional, this reduces to ordinary multiplication of natural numbers.

This section has focused exclusively on the *algebraic* notion of the tensor product of two vector spaces over a field; however, there are also *topological* notions of the tensor product which play an important role in functional analysis in general and in branches of approximation theory in particular. For example, if U and V are *Banach spaces over* $\mathbb{R}$, there are a variety of ways to use the Banach space structures of U and V to induce a Banach space structure on the algebraic tensor product $U \otimes V$. If U and V are *Hilbert spaces over* $\mathbb{R}$, there is in fact a *canonical* way to use the Hilbert space structures of U and V to induce a Hilbert space structure on the algebraic tensor product $U \otimes V$.

Although these topological notions of the tensor product do not play a direct role in this thesis, their general mathematical importance nevertheless shapes the development of the thesis in the following way: By using the *generalized* dual space to develop tensor product theory in a way that is *independent* of any particular notion of the dual space, the thesis develops dual-purpose results; these results can either be used without further modification in purely algebraic applications, or they can be readily equipped with additional topological structure for applications involving functional analysis. In summary, our reliance upon the generalized dual space *minimizes* the amount of labor we must perform while *maximizing* the fruits of that labor in both present and future applications.

Light and Cheney have assembled a body of fundamental results on topological tensor products of two Banach spaces or two Hilbert spaces in [LC85, pp. 1–34]; they emphasize results with a particular relevance to approximation theory, and cite [Sch50] and [DU77] as source material. Readers who are in a big hurry may prefer Cheney's enjoyable survey [Che86, pp. 9–16], which contains a brief summary of the material assembled in the aforementioned research monograph [LC85, pp. 1–34]. Wegge-Olsen's reader-friendly book [WO93, pp. 309–361] contains an extensive appendix which covers tensor products from both the algebraic and topological points of view.[3] Effros and Ruan devote three entire chapters of [ER00, pp. 121–174] to the projective, injective, and Haagerup tensor products of two operator spaces; their book also includes material on the projective and injective tensor products of two Banach spaces. Zemanian's book [Zem72, pp. 34–38, 213–215, Dover] contains a very terse introduction to topological tensor products, but provides a broad context concerned with applications inspired by physical systems. Readers who wish to delve deeply into the topological aspects of tensor product theory will surely find much that is of interest in the various references cited above.

4.2 Tensor Products of Function Spaces Revisited

In the previous section, we developed the tensor product of abstract vector spaces. In Subsection 2.2.2, we had already developed the tensor product of concrete function spaces. The goal of this section is to show that a tensor product space in the sense defined for concrete function spaces is also a tensor product space in the sense defined for abstract vector spaces. In establishing this important connection, we build a major bridge along the thesis highway—a bridge that will allow us to transport the abstract results of this chapter to the many concrete applications of function spaces in subsequent chapters.

Upon comparing Definition 2.3, which defines the tensor product of function spaces, with Definition 4.1, which defines the tensor product of vector spaces, we conclude that the function space definition already has all of the specified properties of the vector space definition except for one: the universal property; thus, to accomplish the goal of this section, it suffices to show that the tensor product of function spaces has the universal property. This requires us to prove that a certain linear map is defined on the tensor product space. We can always define a linear map by prescribing its values on a basis of the tensor product space. In order to construct a suitable basis, we could try to use Hamel Basis Theorem 4.14, which we proved by invoking Cancellation Lemma 4.12. Although this sounds like a viable proof strategy, it has one teeny weeny flaw: *Cancellation Lemma 4.12*

[3] Wegge-Olsen has *already* done the world an enormous favor by using the *universal property* to develop the canonical Hilbert space structure on the algebraic tensor product of two Hilbert spaces. Wegge-Olsen knows a good labor saving device when he sees one!

uses the universal property! Since we are not allowed to *assume* what we are trying to *prove*, we find ourselves confronted with a rather delicate problem.[4] Upon closer examination, however, we will find that this problem already contains within itself the seeds of its own solution.

Precisely how does Cancellation Lemma 4.12 use the universal property? The lemma invokes the universal property *implicitly* in exactly two places: first, where we apply the linear map $I \otimes \psi$ constructed from the linear functional ψ, and second, where we apply the transpose operator T on the tensor product space $U \otimes V$. We had previously used the universal property to construct both $I \otimes \psi$ and T, and therein lies the problem.

What is the solution to this problem? We can salvage our original proof strategy by finding a way to construct the linear maps $I \otimes \psi$ and T without using the universal property. We find the seeds of this solution in something which *vectors* do not generally have, but which *functions* always have: variables! We can construct the linear maps $I \otimes \psi$ and T on the tensor product of function spaces by exploiting the presence of variables. We will now develop each of these two constructions in turn.

Let X and Y be arbitrary sets, and let F be an arbitrary field. Let $U \subset F^X$ and $V \subset F^Y$ be function spaces, and define $W := F^{X \times Y}$ and $\tilde{W} := F^{Y \times X}$. Let $U \otimes V \subset W$ and $V \otimes U \subset \tilde{W}$ denote the tensor products of function spaces.

Note that for all functions $u \in U$ and $v \in V$ and every fixed $x \in X$, the x-section of the dyad $u \otimes v$ satisfies

$$(u \otimes v)_x = u(x) \cdot v \in V.$$

Let $t \in U \otimes V$ be an arbitrary tensor product, and let $\psi \in V^*$ be an arbitrary linear functional. Since t can always be written as a finite sum of dyads, it follows that the x-section of t also satisfies $t_x \in V$ for all $x \in X$; consequently, we can define the parametric extension $\Psi : U \otimes V \to F^X$ of the linear functional ψ by

$$(\Psi t)(x) := \psi(t_x) \quad \text{for all} \quad x \in X.$$

The map Ψ transforms dyads in the following way:

$$\Psi(u \otimes v)(x) := \psi((u \otimes v)_x) = \psi(u(x) \cdot v) = u(x) \cdot \psi(v) = (\psi(v) \cdot u)(x)$$

for all $u \in U$, $v \in V$, and $x \in X$. By eliminating the independent variable x, we can rewrite

[4] That's a "sticky wicket" to all you Anglophones, better known as a "Catch-22" to readers south of the border. Ah, the joys of being bilingual!

the previous equation as

$$\Psi(u \otimes v) = \psi(v) \cdot u \quad \text{for all} \quad u \in U \quad \text{and} \quad v \in V.$$

We recall that the parametric extension of a linear map is always a linear map. The previous equation and the linearity of Ψ together imply that $\Psi t \in U$ for all $t \in U \otimes V$; consequently,

$$\Psi : U \otimes V \to U.$$

We conclude that the parametric extension Ψ has all the essential properties of the map $I \otimes \psi$ previously constructed using the universal property.

We will now define two maps $P : W \to \tilde{W}$ and $\tilde{P} : \tilde{W} \to W$ which transform functions by *permuting* the variables x and y:

$$\begin{aligned} (Pf)(y, x) &:= f(x, y) \quad \text{for all} \quad f \in W \quad \text{and} \quad (y, x) \in Y \times X, \\ (\tilde{P}\tilde{f})(x, y) &:= \tilde{f}(y, x) \quad \text{for all} \quad \tilde{f} \in \tilde{W} \quad \text{and} \quad (x, y) \in X \times Y. \end{aligned}$$

Note that both P and $\tilde{P}$ are linear maps. Furthermore, the composition $\tilde{P}P$ is the identity map on W, and the composition $P\tilde{P}$ is the identity map on $\tilde{W}$; hence, both P and $\tilde{P}$ are bijections with $P^{-1} = \tilde{P}$. In addition, the function spaces W and $\tilde{W}$ are isomorphic.

The map P transforms dyads in the following way:

$$P(u \otimes v)(y, x) := (u \otimes v)(x, y) := u(x)\, v(y) = v(y)\, u(x) =: (v \otimes u)(y, x)$$

for all $u \in U$, $v \in V$, and $(y, x) \in Y \times X$. By eliminating the independent variables y and x, we can rewrite the previous equation as

$$P(u \otimes v) = v \otimes u \quad \text{for all} \quad u \in U \quad \text{and} \quad v \in V.$$

We note that the map P transposes the factors of the dyad $u \otimes v$. Since P is linear, the image of $U \otimes V$ under P satisfies $P(U \otimes V) = V \otimes U$. Since P is a bijection, the function spaces $U \otimes V$ and $V \otimes U$ are isomorphic. We note that P also transposes the factors of the tensor product space $U \otimes V$. We conclude that the restriction $P|U \otimes V$ has all the essential properties of the transpose operator T previously constructed using the universal property.

In the special case of function spaces, we can now define the linear map $I \otimes \psi$ to be the parametric extension Ψ for all $\psi \in V^*$, and we can define the transpose operator T

to be the restriction $P|(U \otimes V)$. If we use these specialized definitions in the proof of Cancellation Lemma 4.12, we immediately obtain the following corollary for the tensor product of functions:

Corollary 4.16 (Cancellation) *Let X and Y be arbitrary sets, and let F be an arbitrary field. Let $U \subset F^X$ and $V \subset F^Y$ be function spaces, and let $\{u_i\}_{i=0}^{n-1} \subset U$ and $\{v_i\}_{i=0}^{n-1} \subset V$, where $n \geq 1$. Assume that*

$$\sum_{i=0}^{n-1} u_i \otimes v_i = 0.$$

The following two results hold independently:

1. *If $\{u_i\}_{i=0}^{n-1}$ is a linearly independent set, then all the elements of $\{v_i\}_{i=0}^{n-1}$ are zero.*

2. *If $\{v_i\}_{i=0}^{n-1}$ is a linearly independent set, then all the elements of $\{u_i\}_{i=0}^{n-1}$ are zero.*

If we now replace Cancellation Lemma 4.12 with Cancellation Corollary 4.16 in the proof of Hamel Basis Theorem 4.14, we immediately obtain the following corollary for the tensor product of function spaces:

Corollary 4.17 (Hamel Basis) *Let X and Y be arbitrary sets, and let F be an arbitrary field. Let $U \subset F^X$ and $V \subset F^Y$ be function spaces, and let $U \otimes V \subset F^{X \times Y}$ denote the tensor product of U and V as function spaces. If $\{u_i\}_{i \in I}$ and $\{v_j\}_{j \in J}$ are Hamel bases of U and V, respectively, then $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$ is a Hamel basis of $U \otimes V$.*

Now that we have dispensed with all the preliminaries, we are ready to prove the main result of this section:

Proposition 4.18 (Universality) *Let X and Y be arbitrary sets, and let F be an arbitrary field. Let $U \subset F^X$ and $V \subset F^Y$ be function spaces, and let $U \otimes V \subset F^{X \times Y}$ denote the tensor product of U and V as function spaces. The function space $U \otimes V$ has the universal property of tensor product spaces.*

Proof. Let W denote any *abstract vector space* over the field F, and let $B : U \times V \to W$ denote any bilinear map. We must construct a linear map $L : U \otimes V \to W$ such that

$$L(u \otimes v) = B(u, v) \quad \text{for all} \quad u \in U \quad \text{and} \quad v \in V.$$

Let $\{u_i\}_{i\in I}$ and $\{v_j\}_{j\in J}$ be Hamel bases of U and V, respectively. Hamel Basis Corollary 4.17 implies that $\{u_i \otimes v_j\}_{(i,j)\in I\times J}$ is a Hamel basis of $U \otimes V$. Define a linear map $L : U \otimes V \to W$ by prescribing its values on the Hamel basis as follows:

$$L(u_i \otimes v_j) := B(u_i, v_j) \quad \text{for all} \quad i \in I \quad \text{and} \quad j \in J.$$

For every $u \in U$ and $v \in V$, there exist nonempty finite index sets $I_0 \subset I$ and $J_0 \subset J$ and sets of coefficients $\{\alpha_i\}_{i\in I_0}, \{\beta_j\}_{j\in J_0} \subset F$ such that

$$u = \sum_{i\in I_0} \alpha_i \cdot u_i \quad \text{and} \quad v = \sum_{j\in J_0} \beta_j \cdot v_j.$$

The bilinearity of $\otimes$ implies that

$$u \otimes v = \sum_{i\in I_0} \sum_{j\in J_0} \alpha_i\, \beta_j \cdot (u_i \otimes v_j).$$

The linearity of L and the bilinearity of B further imply that

$$L(u \otimes v) = \sum_{i\in I_0} \sum_{j\in J_0} \alpha_i\, \beta_j \cdot L(u_i \otimes v_j) = \sum_{i\in I_0} \sum_{j\in J_0} \alpha_i\, \beta_j \cdot B(u_i, v_j) = B(u, v),$$

as desired. ∎

Now that we know the tensor product of function spaces has the universal property, we can conclude that every tensor product of function spaces in the sense of Definition 2.3 is *also* a tensor product of vector spaces in the sense of Definition 4.1. With this wonderful fact at our disposal, we can now reap the enormous benefits of the universal property in all subsequent applications of tensor product methods to function spaces. We will start reaping the many benefits of abstraction in the next section, where our systematic study of the nonuniqueness of tensor product representations formally begins.

4.3 Normal Forms with Respect to the First Factor

Let U and V be vector spaces over a field F. The nonuniqueness of tensor product representations that we observed in examples based on function spaces in Subsection 2.2.2 also occurs in the more abstract setting—every element of the tensor product space $U \otimes V$ can be represented *in more than one way* as an expression of the form

$$\sum_{i=0}^{n-1} u_i \otimes v_i. \tag{4.3}$$

Let E denote the set of all such possible expressions. From a rigorous perspective, it is unclear whether E contains a subset C of *canonical forms* which provides a unique representation for every tensor product; however, E always contains a subset N of *normal forms* which provides a unique representation for the tensor product zero. These normal forms, in turn, allow us to solve the zero equivalence problem for tensor products in the sense of computer algebra.

The **zero equivalence problem for tensor products** is the following *decision problem:* Can we find a *decision algorithm* which determines whether or not a given expression $e \in E$ represents the tensor product zero? If we can find a decision algorithm for *zero equivalence*, we immediately obtain a decision algorithm for *general equivalence* in the following way: To determine whether two distinct expressions $e_1, e_2 \in E$ represent the same tensor product in $U \otimes V$, we simply determine whether their difference $e_1 - e_2$ represents the tensor product zero. The general equivalence problem is the reason why the zero equivalence problem is important.

In order to discuss normal forms for tensor products in a clear and precise way, we now introduce some formalisms from computer algebra, following the treatment in [GCL92, pp. 80–84] with a few minor modifications. Let E_n denote the set of all n-term expressions of the form (4.3). By convention, $E_0 := \{0\}$. We formally define the set of all possible tensor product expressions by

$$E := \bigcup_{n \in \mathbb{N}} E_n.$$

We call E the **set of expressions over** $U \otimes V$. If $e_1, e_2 \in E$, we write $e_1 \simeq e_2$ to indicate that *e_1 and e_2 represent the same tensor product in $U \otimes V$*, and we write $e_1 \cong e_2$ to indicate that *e_1 and e_2 are identical expressions in E*. Clearly, $e_1 \cong e_2$ implies $e_1 \simeq e_2$; however, due to the nonuniqueness of tensor product representations, $e_1 \simeq e_2$ need not imply $e_1 \cong e_2$. For example, if $F \subset \mathbb{C}$ is a subfield, the following relationships hold in the set of expressions over $F[x] \otimes F[y]$:

$$\begin{aligned} 0 &\simeq 1 \otimes (-q) \;+\; (1-2p) \otimes q \;+\; p \otimes (2q) \\ 0 &\not\cong 1 \otimes (-q) \;+\; (1-2p) \otimes q \;+\; p \otimes (2q). \end{aligned}$$

In this example, p and q denote the functions induced by arbitrary polynomials $p(x) \in F[x]$ and $q(y) \in F[y]$.

Both $\simeq$ and $\cong$ are equivalence relations on E. In the terminology of mathematical logic as exemplified in [Sho67, p. 3] and [NS93, p. 5], $\simeq$ describes *semantic* equivalence in E, while $\cong$ describes *syntactic* equivalence in E; the semantics capture the *meaning* of an expression when it is interpreted mathematically as a tensor product, while the syntax captures the *structure* of an expression as a formal string of symbols. From the perspective of abstract algebra, semantic equivalence $\simeq$ is just ordinary equality in the tensor product space $U \otimes V$; in fact, we can think of $U \otimes V$ as the set $E/\simeq$ of all equivalence classes in E with respect to the equivalence relation $\simeq$. The following formal definition reveals the reason for introducing these two equivalence relations on E:

Definition 4.19 *We call $\rho : E \to E$ an* ***idempotent normal function for*** $(E, \simeq, \cong)$ *if ρ can be computed by a finitary algorithm, and the following three properties hold for all expressions $e \in E$:*

1. $\rho(e) \simeq e$,
2. $\rho(e) \cong \rho(0)$ *whenever* $e \simeq 0$, *and*
3. $\rho(\rho(e)) \cong \rho(e)$.

Let $N := \rho(E)$, the range of ρ. If ρ is an idempotent normal function for $(E, \simeq, \cong)$, we call the elements of the set N the ***normal forms for*** $(E, \simeq, \cong)$.

In general, a normal function ρ as defined in [GCL92, pp. 80–84] need *not* be idempotent; in that case, the set of normal forms is defined as the set $\{e \in E : \rho(e) \cong e\}$ of **fixed-points of** ρ with respect to the equivalence relation $\cong$. When ρ is **idempotent** with respect to $\cong$, as defined in Property 3 of Definition 4.19, the set of fixed-points of ρ and the range of ρ are exactly the same; in that case, the two definitions of normal forms are equivalent. We use the range definition here merely because it is simpler than the fixed-point definition.

What does Definition 4.19 say in ordinary words? Property 1 says that the normal function ρ reduces every expression $e \in E$ to a normal form $\rho(e) \in N$ which *represents the same tensor product* as the original expression e. Property 2 says that every expression e which represents the tensor product 0 *has the unique normal form* $\rho(0)$. Properties 1 and

2 together guarantee that the expression e represents the tensor product 0 *if and only if*[5] the normal form $\rho(e)$ and the normal form $\rho(0)$ are identical expressions; hence, the normal function ρ and normal forms N solve the zero equivalence problem for E. Property 3, which is added for simplicity, says that ρ is an idempotent function on the set of expressions E; this guarantees that if we try to further reduce a normal form $\rho(e)$ by applying the normal function ρ again, *the expression $\rho(e)$ remains unaltered.* To understand how we might actually *construct* a function ρ with all these properties, consider the following algorithm specification:

Algorithm 4.20 (DIRECT over U) *Let $\rho_U : E \to E$ denote the function computed by the following five-step algorithm:*

1. ***Dispatch irregularities*** *by screening the input $e \in E$ for the following exceptional cases:*

   (a) *If $e \cong 0$, simply define $\rho_U(e) := 0$ and exit the algorithm.*

   (b) *We may now assume*
   $$e \cong \sum_{i=0}^{n-1} u_i \otimes v_i \tag{4.4}$$
   for some $n \geq 1$. Form the finite-dimensional subspace $\bar{U} := \operatorname{span}\{u_i\}_{i=0}^{n-1}$ and let $m := \dim \bar{U}$; note that $\bar{U} \subset U$ and $m \leq n$.

   (c) *If $m = 0$, then $u_i = 0$ for $0 \leq i \leq n-1$, and $e \simeq 0$. Define $\rho_U(e) := 0$ and exit the algorithm.*

   (d) *If $m = n$, then $\{u_i\}_{i=0}^{n-1}$ is a basis of $\bar{U}$. Let $\bar{u}_i := u_i$ and $\bar{v}_i := v_i$ for $0 \leq i \leq n-1$, and skip to Step 5 of the algorithm.*

2. ***Rewrite*** *the generators $\{u_i\}_{i=0}^{n-1}$ of $\bar{U}$ in terms of a basis $\{\bar{u}_i\}_{i=0}^{m-1} \subset \{u_i\}_{i=0}^{n-1}$ of $\bar{U}$ to obtain*
   $$u_i = \sum_{j=0}^{m-1} a_{ij}\, \bar{u}_j \quad \textit{for} \quad 0 \leq i \leq n-1, \tag{4.5}$$
   and substitute equation (4.5) into the original expression (4.4) to obtain
   $$e \cong \sum_{i=0}^{n-1} \left(\sum_{j=0}^{m-1} a_{ij}\, \bar{u}_j \right) \otimes v_i. \tag{4.6}$$

[5] Property 2 states that $e \simeq 0$ implies $\rho(e) \cong \rho(0)$. Conversely, recall that $\rho(e) \cong \rho(0)$ implies $\rho(e) \simeq \rho(0)$; since $\simeq$ is an equivalence relation, $e \simeq \rho(e) \simeq \rho(0) \simeq 0$ by Property 1, and thus $e \simeq 0$.

3. ***Expand*** *expression (4.6) using the bilinearity of* $\otimes$ *to obtain the linear combination of dyads*
$$e \simeq \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} a_{ij} \left(\bar{u}_j \otimes v_i \right). \tag{4.7}$$

4. ***Collect*** *like terms of expression (4.7) with respect to the basis* $\{\bar{u}_i\}_{i=0}^{m-1}$ *using the bilinearity of* $\otimes$ *to obtain*
$$e \simeq \sum_{j=0}^{m-1} \bar{u}_j \otimes \left(\sum_{i=0}^{n-1} a_{ij}\, v_i \right) \cong \sum_{j=0}^{m-1} \bar{u}_j \otimes \bar{v}_j$$
where
$$\bar{v}_j := \sum_{i=0}^{n-1} a_{ij}\, v_i \quad \textit{for} \quad 0 \le j \le m-1.$$

5. ***Terminate*** *the algorithm by defining* $K := \{k \in \mathbb{N} : 0 \le k \le m-1 \textit{ and } \bar{v}_k \neq 0\}$ *and considering the following two cases:*

   (a) *If* $K = \varnothing$*, then* $e \simeq 0$*. Define* $\rho_U(e) := 0$ *and exit the algorithm.*

   (b) *Since we may now assume* $K \neq \varnothing$*, Cancellation Lemma 4.12 allows us to deduce that* $e \not\simeq 0$*. Define*
   $$\rho_U(e) := \sum_{k \in K} \bar{u}_k \otimes \bar{v}_k$$
   (summing over the index set K *in increasing order) and exit the algorithm.*

The boldfaced words which summarize the five major steps of the algorithm—**dispatch irregularities, rewrite, expand, collect, terminate**—form the acronym **DIRECT**.[6] We call Algorithm 4.20 the **DIRECT algorithm over** U because the algorithm uses a basis $\{\bar{u}_i\}_{i=0}^{m-1}$ of a subspace $\bar{U}$ of the vector space U. The following theorem assures us that the DIRECT algorithm over U actually does exactly what we want it to do:

Theorem 4.21 (Correctness) *The function* ρ_U *calculated by the DIRECT algorithm over* U *is an idempotent normal function for* $(E, \simeq, \cong)$ *in the sense of Definition 4.19.*

Proof. Property 1 of Definition 4.19 holds since the DIRECT algorithm over U transforms the original input $e \in E$ using only transformations which preserve the equivalence of

[6] Coming up with snappy-sounding acronyms is hard work! Reading other people's acronyms, however, is great fun. For endless entertainment with acronyms, go online and consult http://www.acronymfinder.com.

expressions in E with respect to either $\cong$ or $\simeq$. Since equivalence with respect to $\cong$ implies equivalence with respect to $\simeq$, it follows that $\rho_U(e) \simeq e$.

Property 2 holds because the algorithm must terminate in Step 1a, 1c, or 5a if $e \simeq 0$; the algorithm returns $\rho_U(e) := 0$ in each case. Since Step 1a defines $\rho_U(0) := 0$, it follows that $\rho_U(e) \cong \rho_U(0)$ whenever $e \simeq 0$.

Property 3 holds in the case $e \simeq 0$ because $\rho_U(e) := 0$ and $\rho_U(0) := 0$ together imply

$$\rho_U(\rho_U(e)) \cong \rho_U(0) \cong 0 \cong \rho_U(e).$$

All that remains is to show that Property 3 also holds if $e \not\simeq 0$.

If $e \not\simeq 0$, the algorithm must terminate in Step 5b, returning a value expressed in *output notation* as

$$\rho_U(e) := \sum_{k \in K} \bar{u}_k \otimes \bar{v}_k.$$

Note that $K \neq \varnothing$, the set $\{\bar{u}_k\}_{k \in K}$ is linearly independent, and all the elements of $\{\bar{v}_k\}_{k \in K}$ are nonzero. Letting $n := |K|$ and rewriting $\rho_U(e)$ in *input notation* to prepare for a second application of ρ_U yields

$$\rho_U(e) := \sum_{i=0}^{n-1} u_i \otimes v_i$$

where $n \geq 1$, the set $\{u_i\}_{i=0}^{n-1}$ is linearly independent, and all the elements of $\{v_i\}_{i=0}^{n-1}$ are nonzero.

Now consider the calculation of $\rho_U(\rho_U(e))$: Since the set $\{u_i\}_{i=0}^{n-1}$ is linearly independent, Step 1d determines that $m = n$, defines a new $\bar{u}_i := u_i$ and a new $\bar{v}_i := v_i$ for $0 \leq i \leq n-1$, and skips to Step 5. Since all the elements of $\{\bar{v}_k\}_{k=0}^{m-1} = \{v_i\}_{i=0}^{n-1}$ are nonzero, Step 5 defines a new $K := \{0, 1, \ldots, n-1\}$. Since $n \geq 1$, it follows that the new $K \neq \varnothing$. The algorithm therefore terminates in Step 5b, and returns a value satisfying

$$\rho_U(\rho_U(e)) \cong \sum_{k \in K} \bar{u}_k \otimes \bar{v}_k \cong \sum_{k=0}^{n-1} u_k \otimes v_k \cong \rho_U(e).$$

We conclude that Property 3 also holds in the case $e \not\simeq 0$, and thus ρ_U is an idempotent normal function for $(E, \simeq, \cong)$. ■

The previous theorem allows us to make the following definition:

Definition 4.22 *Let $N_U := \rho_U(E)$, the range of ρ_U. By Definition 4.19 and Correctness Theorem 4.21 for the DIRECT algorithm over U, the elements of N_U are normal forms for*

$(E, \simeq, \cong)$. More specifically, we call the elements of N_U the ***normal forms over*** *U.*

Our earlier work leads immediately to the following simple characterization of the normal forms over U:

Corollary 4.23 *The normal forms over U with respect to $U \otimes V$ consist of*

1. *the expression 0, and*
2. *all expressions of the form*
$$\sum_{i=0}^{n-1} u_i \otimes v_i$$
where $n \geq 1$, the set $\{u_i\}_{i=0}^{n-1} \subset U$ is linearly independent, and all the elements of $\{v_i\}_{i=0}^{n-1} \subset V$ are nonzero.

Proof. In the proof of Correctness Theorem 4.21, we showed for all $e \in E$ that the expression $\rho_U(e)$ has the first form if $e \simeq 0$ and the second form if $e \not\simeq 0$; hence, every element in the range of ρ_U must have one of the two forms stated above. Conversely, the same reasoning we used to establish the idempotence of ρ_U in the proof of Correctness Theorem 4.21 also shows that both of these two forms are fixed-points of ρ_U, and thus belong to the range of ρ_U. ■

We will now consider some examples which illustrate how the DIRECT algorithm can be used to reduce a nontrivial representation of the tensor product 0 to the normal form 0. Let $F \subset \mathbb{C}$ be a subfield, let $p(x) \in F[x]$ be *nonconstant* and $q(y) \in F[y]$ be *nonzero*, and let p and q denote the corresponding induced functions. Recall that the tensor product 0 has the following nontrivial representation in $F[x] \otimes F[y]$:

$$1 \otimes (-q) \;+\; (1-2p) \otimes q \;+\; p \otimes (2q). \tag{4.8}$$

Let us apply the DIRECT algorithm over $F[x]$ to reduce expression (4.8) to normal form over $F[x]$. Note that the expression has $n = 3$ terms. Define the subspace $\bar{U} := \mathrm{span}_F\{1, 1-2p, p\}$ of $F[x]$ with $m := \dim_F \bar{U} = 2$, and select the basis $\{1, p\} \subset \{1, 1-2p, p\}$ of $\bar{U}$. Since $0 < m < n$, there are no irregularities to dispatch. Since the generators $\{1, 1-2p, p\}$ are already expressed in terms of the basis $\{1, p\}$, there is nothing to rewrite. Expanding expression (4.8) over $F[x]$ into a linear combination of dyads, we obtain

$$1 \otimes (-q) \;+\; 1 \otimes q \;+\; (-2)(p \otimes q) \;+\; p \otimes (2q). \tag{4.9}$$

Collecting like terms of expression (4.9) with respect to the basis $\{1, p\}$ yields

$$1 \otimes (-q + q) \ + \ p \otimes (-2q + 2q) \quad \cong \quad 1 \otimes 0 \ + \ p \otimes 0. \tag{4.10}$$

Terminating, we obtain the index set $K = \varnothing$, which means that expression (4.10) represents the tensor product 0 and thus has normal form 0 over $F[x]$.

For comparison, let us now apply a variation of the DIRECT algorithm to the original expression

$$1 \otimes (-q) \ + \ (1 - 2p) \otimes q \ + \ p \otimes (2q). \tag{4.11}$$

Finding a basis of a subspace of $F[y]$ instead of $F[x]$ will yield a normal form over $F[y]$ instead of $F[x]$; however, since expression (4.11) represents the tensor product 0, we expect that the resulting normal form over $F[y]$ will still be 0.

As before, we note that expression (4.11) has $n = 3$ terms. Define the subspace $\bar{V} := \mathrm{span}_F\{-q, q, 2q\}$ of $F[y]$ with $m := \dim_F \bar{V} = 1$, and select the basis $\{q\} \subset \{-q, q, 2q\}$ of $\bar{V}$. Since $0 < m < n$, there are no irregularities to dispatch. Since the generators $\{-q, q, 2q\}$ are already expressed in terms of the basis $\{q\}$, there is nothing to rewrite. Expanding expression (4.11) over $F[y]$ into a linear combination of dyads, we obtain

$$(-1)(1 \otimes q) \ + \ (1 - 2p) \otimes q \ + \ 2(p \otimes q). \tag{4.12}$$

Collecting like terms of expression (4.12) with respect to the basis $\{q\}$ yields

$$(-1 + 1 - 2p + 2p) \otimes q \quad \cong \quad 0 \otimes q. \tag{4.13}$$

Terminating, we obtain the index set $K = \varnothing$, which means that expression (4.13) represents the tensor product 0 and thus has normal form 0 over $F[y]$.

We have done the same example two different ways to illustrate informally that the DIRECT algorithm over U and the normal forms over U both have *duals*, namely the DIRECT algorithm over V and the normal forms over V; these duals arise by transposing the *roles* of U and V (but not the *symbols* U and V) while working with expressions over $U \otimes V$. Instead of mindlessly repeating all of our previous definitions and theorems with minor variations, the next section will use the transpose operators T and $\tilde{T}$ to reduce the DIRECT algorithm over V and the normal forms over V to the previously studied case by systematically exploiting the underlying duality.

4.4 Normal Forms with Respect to the Second Factor

Let U and V be vector spaces over a field F. Let T and $\tilde{T}$ denote the transpose operators on the tensor product spaces $U \otimes V$ and $V \otimes U$, respectively, and let E and $\tilde{E}$ denote the sets of expressions over $U \otimes V$ and $V \otimes U$, respectively.

The additivity of T on $U \otimes V$ induces a bijection $\tau : E \to \tilde{E}$ which we define by

$$\tau\left(\sum_{i=0}^{n-1} u_i \otimes v_i\right) := \sum_{i=0}^{n-1} v_i \otimes u_i.$$

When $n = 0$, our convention for the empty sum yields $\tau(0) := \tilde{0}$, where $\tilde{0}$ denotes the zero expression in $\tilde{E}$. The inverse $\tilde{\tau} : \tilde{E} \to E$ is similarly defined. Let $e_i \in E$ be arbitrary and define $\tilde{e}_i \in \tilde{E}$ by $\tilde{e}_i := \tau(e_i)$ for $i = 1, 2$. Note that $e_1 \simeq e_2$ in E if and only if $\tilde{e}_1 \simeq \tilde{e}_2$ in $\tilde{E}$, and $e_1 \cong e_2$ in E if and only if $\tilde{e}_1 \cong \tilde{e}_2$ in $\tilde{E}$; thus, τ and $\tilde{\tau}$ preserve the equivalence relations $\simeq$ and $\cong$. In addition, the well-defined map $\bar{\tau} : (E/\simeq) \to (\tilde{E}/\simeq)$ between sets of equivalence classes recovers the original map $T : U \otimes V \to V \otimes U$ between tensor product spaces, which means that the induced map $\tau : E \to \tilde{E}$ between sets of expressions is a *compatible refinement* of the original map T. This compatibility is the direct result of using the *properties* of T on the space $U \otimes V$ to define the *values* of τ on the elements of E.

Let $\tilde{\rho}_V : \tilde{E} \to \tilde{E}$ denote the function calculated by the DIRECT algorithm over V *with respect to* $V \otimes U$, which is simply Algorithm 4.20 with the *symbols* U and V transposed throughout. By Correctness Theorem 4.21, $\tilde{\rho}_V$ is an idempotent normal function for $(\tilde{E}, \simeq, \cong)$, and $\tilde{N}_V := \tilde{\rho}_V(\tilde{E})$ is the corresponding set of normal forms for $(\tilde{E}, \simeq, \cong)$. We are now ready to define the DIRECT algorithm over V *with respect to* $U \otimes V$, and to investigate its properties:

Algorithm 4.24 (DIRECT over V) *Define the function* $\rho_V : E \to E$ *by*

$$\rho_V := \tilde{\tau} \circ \tilde{\rho}_V \circ \tau.$$

We call Algorithm 4.24 the **DIRECT algorithm over** V because the algorithm makes internal use of a basis of a subspace of the vector space V. The following theorem assures us that the DIRECT algorithm over V actually does exactly what we want it to do:

Theorem 4.25 (Correctness) *The function ρ_V calculated by the DIRECT algorithm over V is an idempotent normal function for $(E, \simeq, \cong)$ in the sense of Definition 4.19.*

Proof. In essence, ρ_V is an idempotent normal function for $(E, \simeq, \cong)$ because $\tilde{\rho}_V$ is an idempotent normal function for $(\tilde{E}, \simeq, \cong)$, and the bijections τ and $\tilde{\tau}$ preserve the equivalence relations $\simeq$ and $\cong$. Here are the details:

Let $e \in E$ be arbitrary and define $\tilde{e} \in \tilde{E}$ by $\tilde{e} := \tau(e)$. Recall that $\tau(0) := \tilde{0}$. Property 1 of Definition 4.19 holds for ρ_V since

$$\rho_V(e) \cong \tilde{\tau}\left(\tilde{\rho}_V\left(\tau(e)\right)\right) \cong \tilde{\tau}\left(\tilde{\rho}_V\left(\tilde{e}\right)\right) \simeq \tilde{\tau}\left(\tilde{e}\right) \cong \tilde{\tau}\left(\tau\left(e\right)\right) \cong e.$$

Property 2 holds for ρ_V because $e \simeq 0$ implies $\tilde{e} \cong \tau(e) \simeq \tau\left(0\right) \cong \tilde{0}$, which further implies

$$\rho_V(e) \cong \tilde{\tau}\left(\tilde{\rho}_V\left(\tau(e)\right)\right) \cong \tilde{\tau}\left(\tilde{\rho}_V\left(\tilde{e}\right)\right) \cong \tilde{\tau}\left(\tilde{\rho}_V\left(\tilde{0}\right)\right) \cong \tilde{\tau}\left(\tilde{\rho}_V\left(\tau\left(0\right)\right)\right) \cong \rho_V(0).$$

Property 3 holds for ρ_V because our definitions imply

$$\rho_V \circ \rho_V = \left(\tilde{\tau} \circ \tilde{\rho}_V \circ \tau\right) \circ \left(\tilde{\tau} \circ \tilde{\rho}_V \circ \tau\right) = \tilde{\tau} \circ \tilde{\rho}_V \circ \tilde{\rho}_V \circ \tau,$$

and thus

$$\rho_V\left(\rho_V(e)\right) \cong \tilde{\tau}\left(\tilde{\rho}_V\left(\tilde{\rho}_V\left(\tau(e)\right)\right)\right) \cong \tilde{\tau}\left(\tilde{\rho}_V\left(\tilde{\rho}_V\left(\tilde{e}\right)\right)\right) \cong \tilde{\tau}\left(\tilde{\rho}_V\left(\tilde{e}\right)\right) \cong \tilde{\tau}\left(\tilde{\rho}_V\left(\tau\left(e\right)\right)\right) \cong \rho_V(e).$$

This completes the proof. ■

The previous theorem allows us to make the following definition:

Definition 4.26 *Let $N_V := \rho_V(E)$, the range of ρ_V. By Definition 4.19 and Correctness Theorem 4.25 for the DIRECT algorithm over V, the elements of N_V are normal forms for $(E, \simeq, \cong)$. More specifically, we call the elements of N_V the **normal forms over** V.*

Our earlier work leads immediately to the following simple characterization of the normal forms over V:

Corollary 4.27 *The normal forms over V with respect to $U \otimes V$ consist of*

1. *the expression* 0*, and*
2. *all expressions of the form*
$$\sum_{i=0}^{n-1} u_i \otimes v_i$$
where $n \geq 1$, the set $\{v_i\}_{i=0}^{n-1} \subset V$ is linearly independent, and all the elements of $\{u_i\}_{i=0}^{n-1} \subset U$ are nonzero.

Proof. Since $\tilde{\rho}_V$ is an idempotent normal function for $(\tilde{E}, \simeq, \cong)$, Corollary 4.23 yields the following characterization of the normal forms over V *with respect to* $V \otimes U$:

$$\tilde{N}_V := \tilde{\rho}_V(\tilde{E}) = \{\tilde{0}\} \cup \left\{ \sum_{i=0}^{n-1} v_i \otimes u_i : n \geq 1 \text{ with } \{v_i\}_{i=0}^{n-1} \text{ and } \{u_i\}_{i=0}^{n-1} \text{ as specified} \right\}.$$

The qualification "as specified" means that each set $\{v_i\}_{i=0}^{n-1}$ is linearly independent and all the elements of each set $\{u_i\}_{i=0}^{n-1}$ are nonzero. This characterization and our definitions imply

$$\begin{aligned} N_V &:= \rho_V(E) = \tilde{\tau}\left(\tilde{\rho}_V\left(\tau\left(E\right)\right)\right) = \tilde{\tau}\left(\tilde{\rho}_V\left(\tilde{E}\right)\right) = \tilde{\tau}\left(\tilde{N}_V\right) \\ &= \{0\} \cup \left\{ \sum_{i=0}^{n-1} u_i \otimes v_i : n \geq 1 \text{ with } \{v_i\}_{i=0}^{n-1} \text{ and } \{u_i\}_{i=0}^{n-1} \text{ as specified} \right\}. \end{aligned}$$

The qualification "as specified" has the same meaning as before, which proves the desired characterization of the normal forms over V *with respect to* $U \otimes V$. ■

To summarize, in the previous section, we developed an idempotent normal function ρ_U with normal forms N_U. In this section, we used the *duality of the factors* in the tensor product space $U \otimes V$ to develop an idempotent normal function ρ_V with normal forms N_V. In the next section, we will use these two idempotent normal functions and sets of normal forms *in combination* to create a new collection of normal forms that offers a variety of theoretical and practical advantages.

4.5 Binormal Forms for Tensor Product Expressions

Now that we have developed the idempotent normal functions ρ_U and ρ_V and corresponding sets of normal forms N_U and N_V for expressions over the tensor product space $U \otimes V$, we will show that $\rho_U \circ \rho_V$ and $\rho_V \circ \rho_U$ are idempotent normal functions with the same set of normal forms $N_U \cap N_V$. These are the most useful normal forms in applications of tensor products to interpolation and approximation. We will begin by proving the following invariance result, and will follow this with a rapid development of its consequences:

Theorem 4.28 (Mutual Invariance)

$$\rho_U\left(N_V\right) \subset N_V \quad \textit{and} \quad \rho_V\left(N_U\right) \subset N_U.$$

Proof. Our strategy is to prove $\rho_U(N_V) \subset N_V$ first and then show that $\rho_V(N_U) \subset N_U$ follows by duality. The characterization of N_V by Corollary 4.27 tells us that there are two cases to consider. The trivial case $0 \in N_V$ immediately yields $\rho_U(0) := 0 \in N_V$. The nontrivial case begins by considering an expression $e \in N_V$ of the form

$$e := \sum_{i=0}^{n-1} u_i \otimes v_i$$

where $n \geq 1$, the set $\{v_i\}_{i=0}^{n-1}$ is linearly independent, and all the elements of $\{u_i\}_{i=0}^{n-1}$ are nonzero. Since $e \not\simeq 0$ in this case, the DIRECT algorithm over U returns the value

$$\rho_U(e) := \sum_{k \in K} \bar{u}_k \otimes \bar{v}_k \tag{4.14}$$

where $K \neq \varnothing$, the set $\{\bar{u}_k\}_{k\in K}$ is linearly independent, and all the elements of $\{\bar{v}_k\}_{k\in K}$ are nonzero.

We know from the internal specification of the DIRECT algorithm over U that

$$\begin{aligned} u_i &= \sum_{j=0}^{m-1} a_{ij}\, \bar{u}_j \quad \text{for} \quad 0 \leq i \leq n-1 && (4.15)\\ \bar{v}_j &:= \sum_{i=0}^{n-1} a_{ij}\, v_i \quad \text{for} \quad 0 \leq j \leq m-1 && (4.16) \end{aligned}$$

where $1 \leq m \leq n$. The crux of the nontrivial case is to show that $\{\bar{v}_j\}_{j=0}^{m-1}$ is a linearly independent set. Let $\{c_j\}_{j=0}^{m-1} \subset F$ be arbitrary and suppose that

$$\sum_{j=0}^{m-1} c_j\, \bar{v}_j = 0. \tag{4.17}$$

We must show that the coefficients $c_j = 0$ for $0 \leq j \leq m-1$. Substituting equation (4.16) into equation (4.17) yields

$$\sum_{j=0}^{m-1} c_j \left(\sum_{i=0}^{n-1} a_{ij}\, v_i \right) = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{m-1} a_{ij}\, c_j \right) v_i = 0,$$

which in turn implies

$$\sum_{j=0}^{m-1} a_{ij}\, c_j = 0 \quad \text{for} \quad 0 \le i \le n-1, \tag{4.18}$$

since $\{v_i\}_{i=0}^{n-1}$ is a linearly independent set by assumption.

Define the $n \times m$ matrix $A := [a_{ij}]$ and the m-dimensional column vector $c := [c_j]$, and rewrite the linear system (4.18) in matrix form as $A \cdot c = 0$. Recall that the DIRECT algorithm over U defines $\{\bar{u}_j\}_{j=0}^{m-1}$ to be a basis of $\operatorname{span}_F \{u_i\}_{i=0}^{n-1}$. Equation (4.15) in turn defines A as the matrix whose entries express the elements of $\{u_i\}_{i=0}^{n-1}$ as linear combinations of the elements of the basis $\{\bar{u}_j\}_{j=0}^{m-1}$. Since the internal specification of the DIRECT algorithm over U stipulates that $\{\bar{u}_j\}_{j=0}^{m-1}$ is a *subset* of $\{u_i\}_{i=0}^{n-1}$, the $n \times m$ matrix A contains an $m \times m$ submatrix which is a permutation of the $m \times m$ identity matrix. This implies that $\operatorname{rank} A = m$, or equivalently, that the m columns of A are linearly independent; consequently, the linear system $A \cdot c = 0$ has only the trivial solution $c = 0$, which shows that $\{\bar{v}_j\}_{j=0}^{m-1}$ is a linearly independent set.

Since the set $\{\bar{v}_j\}_{j=0}^{m-1}$ is linearly independent, it follows that

$$K := \{k \in \mathbb{N} : 0 \le k \le m-1 \text{ and } \bar{v}_k \neq 0\} = \{0, 1, \ldots, m-1\}.$$

Summing explicitly over this index set K in equation (4.14) yields

$$\rho_U(e) \cong \sum_{k \in K} \bar{u}_k \otimes \bar{v}_k \cong \sum_{k=0}^{m-1} \bar{u}_k \otimes \bar{v}_k$$

where $m \ge 1$, and both $\{\bar{u}_k\}_{k=0}^{m-1}$ and $\{\bar{v}_k\}_{k=0}^{m-1}$ are linearly independent sets. We conclude from the characterization of N_V by Corollary 4.27 that $\rho_U(e) \in N_V$, as desired.

We have proven that $\rho_U(N_V) \subset N_V$ *with respect to* $U \otimes V$; transposing the *symbols* U and V and adding the appropriate tildes immediately yields $\tilde{\rho}_V\left(\tilde{N}_U\right) \subset \tilde{N}_U$ *with respect to* $V \otimes U$, where $\tilde{N}_U := \tilde{\rho}_U(\tilde{E})$. From the characterization of N_U by Corollary 4.23 and the characterization of $\tilde{N}_U$ by Corollary 4.27, we see that transposition yields $\tau(N_U) = \tilde{N}_U$. Recall that the DIRECT algorithm over V defines $\rho_V := \tilde{\tau} \circ \tilde{\rho}_V \circ \tau$, and consider

$$\rho_V(N_U) = \tilde{\tau}(\tilde{\rho}_V(\tau(N_U))) = \tilde{\tau}\left(\tilde{\rho}_V\left(\tilde{N}_U\right)\right) \subset \tilde{\tau}\left(\tilde{N}_U\right) = \tilde{\tau}(\tau(N_U)) = N_U.$$

This shows that $\rho_V(N_U) \subset N_U$, as desired. ■

Corollary 4.29 *The functions $\rho_U \circ \rho_V$ and $\rho_V \circ \rho_U$ have the same range:*

$$(\rho_U \circ \rho_V)(E) = N_U \cap N_V = (\rho_V \circ \rho_U)(E).$$

Proof. Mutual Invariance Theorem 4.28 implies that

$$\rho_U(\rho_V(E)) = \rho_U(N_V) \subset N_V \quad \text{and} \quad \rho_V(\rho_U(E)) = \rho_V(N_U) \subset N_U.$$

By definition of the sets of normal forms, we also have

$$\rho_U(\rho_V(E)) \subset \rho_U(E) =: N_U \quad \text{and} \quad \rho_V(\rho_U(E)) \subset \rho_V(E) =: N_V.$$

Together, these results yield the inclusions

$$\rho_U(\rho_V(E)) \subset N_U \cap N_V \quad \text{and} \quad \rho_V(\rho_U(E)) \subset N_U \cap N_V.$$

To show the opposite inclusions, recall that ρ_U and ρ_V are both idempotent, which means that each of these functions is the identity map on its own range; consequently,

$$\begin{aligned} \rho_U(\rho_V(E)) &\supset \rho_U(\rho_V(N_U \cap N_V)) = \rho_U(N_U \cap N_V) = N_U \cap N_V \\ \rho_V(\rho_U(E)) &\supset \rho_V(\rho_U(N_U \cap N_V)) = \rho_V(N_U \cap N_V) = N_U \cap N_V. \end{aligned}$$

The original inclusions and their opposites together imply that

$$\rho_U(\rho_V(E)) = N_U \cap N_V \quad \text{and} \quad \rho_V(\rho_U(E)) = N_U \cap N_V,$$

which completes the proof. ■

Corollary 4.30 *The functions $\rho_U \circ \rho_V$ and $\rho_V \circ \rho_U$ are idempotent normal functions for $(E, \simeq, \cong)$ in the sense of Definition 4.19.*

Proof. We will prove the result for both functions at once by letting $\rho_1, \rho_2 \in \{\rho_U, \rho_V\}$ and assuming $\rho_1 \neq \rho_2$. In essence, $\rho_1 \circ \rho_2$ is an idempotent normal function for $(E, \simeq, \cong)$ because ρ_1 and ρ_2 are idempotent normal functions for $(E, \simeq, \cong)$ with the additional property

$$(\rho_1 \circ \rho_2)(E) = N_U \cap N_V$$

by Corollary 4.29. Here are the details:

Let $e \in E$. Property 1 of Definition 4.19 holds for $\rho_1 \circ \rho_2$ because $\rho_1(\rho_2(e)) \simeq \rho_2(e) \simeq e$. Property 2 holds for $\rho_1 \circ \rho_2$ because $e \simeq 0$ implies $\rho_2(e) \cong \rho_2(0)$, which further implies $\rho_1(\rho_2(e)) \cong \rho_1(\rho_2(0))$. Property 3 holds for $\rho_1 \circ \rho_2$ because the idempotence of ρ_1 and ρ_2 implies that $\rho_1(\rho_2(e)) \in N_U \cap N_V$ is a fixed-point of both functions; consequently,

$$\rho_1(\rho_2(\rho_1(\rho_2(e)))) \cong \rho_1(\rho_1(\rho_2(e))) \cong \rho_1(\rho_2(e)),$$

as desired. ∎

Definition 4.31 *Let $B := N_U \cap N_V$, the range of both $\rho_U \circ \rho_V$ and $\rho_V \circ \rho_U$ by Corollary 4.29. By Definition 4.19 and Corollary 4.30, the elements of B are normal forms for $(E, \simeq, \cong)$. More specifically, we call the elements of B the **binormal forms over** $U \otimes V$.*

Corollary 4.32 *The binormal forms over $U \otimes V$ consist of*

1. *the expression 0, and*
2. *all expressions of the form*
$$\sum_{i=0}^{n-1} u_i \otimes v_i$$
where $n \geq 1$, and the sets $\{u_i\}_{i=0}^{n-1} \subset U$ and $\{v_i\}_{i=0}^{n-1} \subset V$ are linearly independent.

Proof. This characterization of $B := N_U \cap N_V$ follows immediately from the previous characterizations of N_U and N_V by Corollaries 4.23 and 4.27. ∎

We now know that $\rho_U \circ \rho_V$ and $\rho_V \circ \rho_U$ are idempotent normal functions with the same domain E and the same range B. With so many properties in common, is it possible that $\rho_U \circ \rho_V$ and $\rho_V \circ \rho_U$ are in fact the same function? Equivalently, do the functions ρ_U and ρ_V commute? The following example suggests that the answer to these flirtatiously inviting questions is, of course, "No! Not on your life!! How could you even think such a thing?!?!"[7]

Example 4.33 (Noncommutativity) *Let $F \subset \mathbb{C}$ be a subfield. Define the monomials $p_i(x) := x^i$ and $q_j(y) := y^j$, and let p_i and q_j denote the corresponding induced functions. We will reduce the expression*

$$e \quad := \quad p_0 \otimes q_0 \;+\; p_0 \otimes q_1 \;+\; p_1 \otimes q_0$$

[7]Please, don't apologize—your questions were perfectly reasonable! *Counselling for traumatized readers will be provided at the thesis defense.* Tell them Sigmund sent you, and ask for the group discount.

to binormal form over $F[x] \otimes F[y]$ *in two different ways. First, we apply the DIRECT algorithm over* $F[x]$ *to the expression* e *using the basis* $\{p_0, p_1\}$ *of the subspace* $\mathrm{span}_F\{p_0, p_0, p_1\}$ *to obtain*

$$\rho_{F[x]}(e) \quad := \quad p_0 \otimes (q_0 + q_1) \ + \ p_1 \otimes q_0.$$

Since this expression is already a binormal form, we have

$$\left(\rho_{F[y]} \circ \rho_{F[x]}\right)(e) \quad := \quad p_0 \otimes (q_0 + q_1) \ + \ p_1 \otimes q_0.$$

Second, we apply the DIRECT algorithm over $F[y]$ *to the expression* e *using the basis* $\{q_0, q_1\}$ *of the subspace* $\mathrm{span}_F\{q_0, q_1, q_0\}$ *to obtain*

$$\rho_{F[y]}(e) \quad := \quad (p_0 + p_1) \otimes q_0 \ + \ p_0 \otimes q_1.$$

Since this expression is already a binormal form, we have

$$\left(\rho_{F[x]} \circ \rho_{F[y]}\right)(e) \quad := \quad (p_0 + p_1) \otimes q_0 \ + \ p_0 \otimes q_1.$$

Because these two binormal forms are distinct expressions, it follows that

$$\left(\rho_{F[x]} \circ \rho_{F[y]}\right)(e) \neq \left(\rho_{F[y]} \circ \rho_{F[x]}\right)(e),$$

and we conclude that the functions $\rho_{F[x]}$ *and* $\rho_{F[y]}$ *do not commute.*

Let $\rho \in \{\rho_U, \ \rho_V, \ \rho_U \circ \rho_V, \ \rho_V \circ \rho_U\}$ be any one of the four different idempotent normal functions we have developed. Recall that ρ gives us a decision algorithm for zero equivalence, which in turn yields the following decision algorithm for general equivalence: Given arbitrary expressions $e_1, e_2 \in E$, the equivalence $e_1 \simeq e_2$ holds if and only if $\rho(e_1 - e_2) \cong \rho(0) := 0$. If, however, we are given *binormal forms* $b_1, b_2 \in B$ instead of arbitrary expressions, *calculating a new normal form* $\rho(b_1 - b_2)$ *is not the best way to determine whether* $b_1 \simeq b_2$. In the next section, we will develop a better approach *which exploits the special structure that the expressions* $b_1, b_2 \in B$ *already possess as binormal forms.*

4.6 Invariants for Deciding Equivalence

If we can find an *easily calculated function* f on the set of binormal forms B with the property that $f(b_1) = f(b_2)$ whenever $b_1 \simeq b_2$, the invariance of f will provide a *partial criterion for distinguishing between nonequivalent binormal forms*—for if $f(b_1) \neq f(b_2)$, it must follow that $b_1 \not\simeq b_2$. If, in addition, $f(b_1) \neq f(b_2)$ whenever $b_1 \not\simeq b_2$, the invariance of f will provide a *complete criterion for distinguishing between nonequivalent binormal forms*. In that case, the complete invariance of f will yield the following new-and-improved decision algorithm: Given binormal forms $b_1, b_2 \in B$, the equivalence $b_1 \simeq b_2$ holds if and only if $f(b_1) = f(b_2)$.

The following definition formalizes these notions of invariance in the more general context of the set of tensor product expressions E, and constructs six different functions with invariance properties on E. After posing this general definition, we shall discuss the advantages of applying these invariance principles to the subset of binormal forms B.

Definition 4.34 (Invariants of Expressions) *Let E denote the set of expressions over $U \otimes V$, and let S be an arbitrary set. We say that a function $f : E \to S$ is an* ***invariant of*** *E if for all $e_1, e_2 \in E$, the equivalence $e_1 \simeq e_2$ implies $f(e_1) = f(e_2)$. (Note that every invariant $f : E \to S$ induces a well-defined function $\bar{f} : (E/\simeq) \to S$ on the set of equivalence classes.) We say that a function $f : E \to S$ is a* ***complete invariant of*** *E if for all $e_1, e_2 \in E$, the equivalence $e_1 \simeq e_2$ holds if and only if $f(e_1) = f(e_2)$. (Note that every complete invariant of E is automatically an invariant of E.)*

Let the arbitrary expression

$$e := \sum_{i=0}^{n-1} u_i \otimes v_i$$

over $U \otimes V$ have $n \geq 0$ terms, and note that the trivial expression $e := 0$ arises from our convention for the empty sum in the special case $n := 0$. We now define the following six invariants of E, which fall naturally into two groups—the primary invariants and the secondary invariants:

1. *Let $U^\dagger$ and $V^\dagger$ denote generalized dual spaces of U and V, respectively. We define the* ***primary invariants*** *as follows:*

   (a) *The* ***linear representation of*** *e* ***into*** *U, denoted by $\mathrm{rep}_U\, e$, is the finite-rank linear transformation*

   $$\mathrm{rep}_U\, e \in \mathrm{Hom}_F(V^\dagger, U)$$

given by

$$(\mathrm{rep}_U\, e)\,(\psi) := (I \otimes \psi)(e) = \sum_{i=0}^{n-1} \psi(v_i) \cdot u_i \quad \textit{for all} \quad \psi \in V^\dagger .$$

(b) *The* ***linear representation of*** e ***into*** V*, denoted by* $\mathrm{rep}_V\, e$*, is the finite-rank linear transformation*

$$\mathrm{rep}_V\, e \in \mathrm{Hom}_F(U^\dagger, V)$$

given by

$$(\mathrm{rep}_V\, e)\,(\phi) := (\phi \otimes I)(e) = \sum_{i=0}^{n-1} \phi(u_i) \cdot v_i \quad \textit{for all} \quad \phi \in U^\dagger .$$

2. *The* ***secondary invariants*** *are derived from the primary invariants as follows:*

   (a) *The* ***range of*** e ***in*** U*, denoted by* $\mathrm{ran}_U\, e$*, is the finite-dimensional subspace of* U *given by*

   $$\mathrm{ran}_U\, e := \mathrm{ran}\,(\mathrm{rep}_U\, e)\,.$$

   (b) *The* ***range of*** e ***in*** V*, denoted by* $\mathrm{ran}_V\, e$*, is the finite-dimensional subspace of* V *given by*

   $$\mathrm{ran}_V\, e := \mathrm{ran}\,(\mathrm{rep}_V\, e)\,.$$

   (c) *The* ***rank of*** e ***in*** U*, denoted by* $\mathrm{rank}_U\, e$*, is the natural number given by*

   $$\mathrm{rank}_U\, e := \mathrm{rank}\,(\mathrm{rep}_U\, e)\,.$$

   (d) *The* ***rank of*** e ***in*** V*, denoted by* $\mathrm{rank}_V\, e$*, is the natural number given by*

   $$\mathrm{rank}_V\, e := \mathrm{rank}\,(\mathrm{rep}_V\, e)\,.$$

In the case of the trivial expression $e := 0$ *with* $n := 0$ *terms, our convention for the empty sum yields* $\mathrm{rep}_U\, 0 := 0$ *and* $\mathrm{rep}_V\, 0 := 0$ *for the primary invariants, from which we obtain* $\mathrm{ran}_U\, 0 := \{0\}$, $\mathrm{ran}_V\, 0 := \{0\}$, $\mathrm{rank}_U\, 0 := 0$, *and* $\mathrm{rank}_V\, 0 := 0$ *for the secondary invariants.*

There are a number of aspects of Definition 4.34 that require immediate comment. *We* *must, of course, actually prove that the two primary functions*

$$\operatorname{rep}_U : E \to \operatorname{Hom}_F(V^\dagger, U) \quad \textit{and} \quad \operatorname{rep}_V : E \to \operatorname{Hom}_F(U^\dagger, V)$$

and the four secondary functions

$$\operatorname{ran}_U : E \to 2^U, \quad \operatorname{ran}_V : E \to 2^V, \quad \operatorname{rank}_U : E \to \mathbb{N}, \quad \textit{and} \quad \operatorname{rank}_V : E \to \mathbb{N}$$

are invariants of E. Once we have established that the primary function rep_U is an invariant of E, it will follow by duality that the primary function rep_V is an invariant of E as well. Since the range and the rank are well-defined for any linear transformation, and since any function of an invariant is automatically an invariant, it will follow immediately that the secondary functions ran_U, ran_V, rank_U, and rank_V are invariants of E.

In order to be useful, the invariants on E must be *easy to calculate.* The following theorem—*which we state without proof*—provides the means to easily calculate the secondary invariants when they are restricted to the subset of binormal forms B:

Theorem 4.35 (Secondary Invariant Calculation) *Let the arbitrary binormal form*

$$b := \sum_{i=0}^{n-1} u_i \otimes v_i$$

over $U \otimes V$ *have* $n \geq 0$ *terms, and note that the trivial binormal form* $b := 0$ *arises from our convention for the empty sum in the special case* $n := 0$. *We can calculate the secondary invariants of* b *as follows:*

$$\begin{aligned} \operatorname{ran}_U b &= \operatorname{span}_F \{u_i\}_{i=0}^{n-1} \\ \operatorname{ran}_V b &= \operatorname{span}_F \{v_i\}_{i=0}^{n-1} \\ \operatorname{rank}_U b &= n \\ \operatorname{rank}_V b &= n. \end{aligned}$$

In the case of the trivial binormal form $b := 0$ *with* $n := 0$ *terms, the above formulas yield the results of Definition 4.34 by the convention* $\operatorname{span}_F \varnothing = \{0\}$.

Corollary 4.36 *The invariants* rank_U *and* rank_V *are identical as functions on* E.

Proof. Let $e \in E$ be an arbitrary expression, let $b := (\rho_U \circ \rho_V)(e)$ be a binormal form representation of e, and let n denote the number of terms in b. Since $b \simeq e$, and rank_U and

rank_V are both invariants of E, we obtain

$$\operatorname{rank}_U e = \operatorname{rank}_U b = n = \operatorname{rank}_V b = \operatorname{rank}_V e$$

by Secondary Invariant Calculation Theorem 4.35. ■

Definition 4.37 *Since* rank_U *and* rank_V *are identical invariants of E by Corollary 4.36, we can drop the U and V subscripts and simply write* $\operatorname{rank} := \operatorname{rank}_U = \operatorname{rank}_V$. *Given an expression $e \in E$, we will call* $\operatorname{rank} e$ *simply the* ***rank of*** *e from now on.*

Note that we now have two different notions of rank: Definition 2.9 defines the rank for any tensor product $t \in U \otimes V$, and Definition 4.37 defines the rank for any expression $e \in E$. The following theorem explains how these two different notions of rank are related:

Theorem 4.38 (Binormal Form Minimality) *Every binormal form $b \in B$ which represents the tensor product $t \in U \otimes V$ has* $\operatorname{rank} b = \operatorname{rank} t$ *terms.*

Proof. Recall that $\operatorname{rank} t$ is defined as the minimal number of terms among all expressions $e \in E$ which represent t. Subsection 2.2.2 cites an argument of Cheney in [Che86, p. 11] which shows, in the terminology of this thesis, that any expression which represents t and has the minimal number of terms, $\operatorname{rank} t$, must be a binormal form; since at least one such expression is guaranteed to exist, let us denote it by $b_0 \in B$. Secondary Invariant Calculation Theorem 4.35 tells us that the number of terms, $\operatorname{rank} t$, in the binormal form b_0 satisfies $\operatorname{rank} b_0 = \operatorname{rank} t$. If another binormal form $b \in B$ also represents the tensor product t, then $b \simeq b_0$ by the definition of equivalence. Since the rank function is an invariant of E, it follows that $\operatorname{rank} b = \operatorname{rank} b_0 = \operatorname{rank} t$, as desired. ■

This theorem tells us that binormal forms are the optimal representations for tensor products since all such representations have the smallest possible number of terms. The theorem also yields a three-step algorithm for calculating the rank of a tensor product:

Algorithm 4.39 (Rank Calculation) *Given any $t \in U \otimes V$, we can calculate* $\operatorname{rank} t \in \mathbb{N}$ *as follows:*

1. *Let $e \in E$ be any expression which represents the tensor product t.*
2. *Find a binormal form $b \in B$ which represents t by calculating $b := (\rho_U \circ \rho_V)(e)$.*
3. *Count the number of terms in b and return the result, which equals* $\operatorname{rank} t$.

We state without proof that each of the two primary invariants is in fact a complete invariant of E. Although *none* of the secondary invariants are complete invariants of E, they are nevertheless useful. Thanks to Secondary Invariant Calculation Theorem 4.35, it is sometimes much easier to use a secondary invariant than a primary invariant to show that two binormal forms are *not* equivalent. These ideas are summarized in the following specification of a four-step algorithm for deciding the equivalence of binormal forms:

Algorithm 4.40 (Binormal Form Equivalence) *Let B denote the set of binormal forms over $U \otimes V$. Given arbitrary $b_1, b_2 \in B$, we can determine whether $b_1 \simeq b_2$ or $b_1 \not\simeq b_2$ by performing the following sequence of tests (arranged in order of increasing complexity):*

1. *If* $\operatorname{rank} b_1 \neq \operatorname{rank} b_2$, *then exit the algorithm and return* $b_1 \not\simeq b_2$.

2. *If* $\operatorname{ran}_U b_1 \neq \operatorname{ran}_U b_2$, *then exit the algorithm and return* $b_1 \not\simeq b_2$.

3. *If* $\operatorname{ran}_V b_1 \neq \operatorname{ran}_V b_2$, *then exit the algorithm and return* $b_1 \not\simeq b_2$.

4. *If* $\operatorname{rep}_V b_1 \neq \operatorname{rep}_V b_2$, *then return* $b_1 \not\simeq b_2$, *otherwise return* $b_1 \simeq b_2$.

The following example demonstrates how the primary invariant rep_V distinguishes between nonequivalent binormal forms even when all the secondary invariants fail to do so:

Example 4.41 *Let $F \subset \mathbb{C}$ be a subfield. Define the monomials $p_i(x) := x^i$ and $q_j(y) := y^j$, and let p_i and q_j denote the corresponding induced functions. Consider the following two binormal forms over $F[x] \otimes F[y]$:*

$$b_1 \quad := \quad p_0 \otimes q_0 \;+\; p_1 \otimes q_1 \qquad \textit{and} \qquad b_2 \quad := \quad p_0 \otimes q_1 \;+\; p_1 \otimes q_0.$$

Reducing $b_1 - b_2$ to normal form over $F[x]$ yields

$$\rho_{F[x]}(b_1 - b_2) \quad := \quad p_0 \otimes (q_0 - q_1) \;+\; p_1 \otimes (q_1 - q_0) \quad \not\cong \quad 0,$$

which implies that $b_1 \not\simeq b_2$. The secondary invariants fail to distinguish b_1 from b_2 since

$$\operatorname{rank} b_1 := 2 =: \operatorname{rank} b_2$$
$$\operatorname{ran}_{F[x]} b_1 := \operatorname{span}_F\{p_0, p_1\} =: \operatorname{ran}_{F[x]} b_2$$
$$\operatorname{ran}_{F[y]} b_1 := \operatorname{span}_F\{q_0, q_1\} = \operatorname{span}_F\{q_1, q_0\} =: \operatorname{ran}_{F[y]} b_2.$$

To see how the primary invariant $\mathrm{rep}_{F[y]}$ *distinguishes* b_1 *from* b_2*, define a specific linear functional* $\phi \in F[x]^*$ *by prescribing its values on the basis* $\{p_i\}_{i\in\mathbb{N}}$ *of* $F[x]$ *as follows:*

$$\phi(p_i) := \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

Now calculate the following values in $F[y]$*:*

$$\begin{aligned} \left(\mathrm{rep}_{F[y]}\, b_1\right)(\phi) &:= \phi(p_0)\, q_0 + \phi(p_1)\, q_1 = q_0 \\ \left(\mathrm{rep}_{F[y]}\, b_2\right)(\phi) &:= \phi(p_0)\, q_1 + \phi(p_1)\, q_0 = q_1. \end{aligned}$$

Since $q_0 \neq q_1$*, it follows that* $\mathrm{rep}_{F[y]}\, b_1 \neq \mathrm{rep}_{F[y]}\, b_2$*, which implies that* $b_1 \not\simeq b_2$*, as desired.*

The following example demonstrates how to use the primary invariant rep_V to show that two distinct binormal forms are equivalent:

Example 4.42 *Let* $F \subset \mathbb{C}$ *be a subfield. Define the monomials* $p_i(x) := x^i$ *and* $q_j(y) := y^j$*, and let* p_i *and* q_j *denote the corresponding induced functions. In Example 4.33, we showed that the expression*

$$e \quad := \quad p_0 \otimes q_0 \;+\; p_0 \otimes q_1 \;+\; p_1 \otimes q_0$$

over $F[x] \otimes F[y]$ *is equivalent to the distinct binormal forms*

$$b_1 \quad := \quad p_0 \otimes (q_0 + q_1) \;+\; p_1 \otimes q_0 \qquad \text{and} \qquad b_2 \quad := \quad (p_0 + p_1) \otimes q_0 \;+\; p_0 \otimes q_1.$$

Since $e \simeq b_1$ *and* $e \simeq b_2$*, it follows that* $b_1 \simeq b_2$*. The complete invariant* $\mathrm{rep}_{F[y]}$ *provides a second proof that* $b_1 \simeq b_2$ *as follows: For all linear functionals* $\phi \in F[x]^*$*, we have*

$$\begin{aligned} \left(\mathrm{rep}_{F[y]}\, b_1\right)(\phi) &:= \phi(p_0)\,(q_0 + q_1) + \phi(p_1)\, q_0 \\ &= (\phi(p_0) + \phi(p_1))\, q_0 + \phi(p_0)\, q_1 \\ &= \phi(p_0 + p_1)\, q_0 + \phi(p_0)\, q_1 \\ &=: \left(\mathrm{rep}_{F[y]}\, b_2\right)(\phi). \end{aligned}$$

This shows that $\mathrm{rep}_{F[y]}\, b_1 = \mathrm{rep}_{F[y]}\, b_2$*, which implies that* $b_1 \simeq b_2$*, as desired.*

Since rep_V is a complete invariant of E, we could attempt to decide whether $e_1 \simeq e_2$ for *arbitrary expressions* $e_1, e_2 \in E$ by testing whether $\mathrm{rep}_V\, e_1 = \mathrm{rep}_V\, e_2$, or equivalently, by testing whether $\mathrm{rep}_V(e_1 - e_2) = \mathrm{rep}_V\, 0 := 0$. Is this a good idea? The steps used to determine whether $\mathrm{rep}_V(e_1 - e_2) = 0$ are essentially the same as the steps used to determine whether $\rho_V(e_1 - e_2) \cong 0$; both involve one iteration of the rewrite-expand-collect process over the factor V.

Remark 4.43 *We conclude that the invariants of E do not actually become labor-saving devices for deciding equivalence until they are applied to the subset of binormal forms B; the savings come from the possibility of quickly determining that $b_1, b_2 \in B$ satisfy $b_1 \not\simeq b_2$ via Secondary Invariant Calculation Theorem 4.35. In summary, Binormal Form Equivalence Algorithm 4.40 may cost much less than the DIRECT algorithm, but it never costs much more—even in the worst case.*

How did the primary invariants rep_U and rep_V get their names? The author commandeered the terminology "linear representation" from the representation theory of associative algebras, as found in [Pie82, pp. 7, 80]. To understand the sense in which this terminology is applicable, recall that we may identify tensor products in the space $U \otimes V$ with equivalence classes in the set $E/\simeq$. If the tensor product $t \in U \otimes V$ is represented by an expression $e \in E$, denote the corresponding equivalence class by $\bar{e} \in (E/\simeq)$, and define the bijection $\varepsilon : U \otimes V \to (E/\simeq)$ by $\varepsilon(t) := \bar{e}$. Since rep_V is an invariant of E, the map $\overline{\mathrm{rep}}_V : (E/\simeq) \to \mathrm{Hom}_F(U^\dagger, V)$ given by $\overline{\mathrm{rep}}_V\, \bar{e} := \mathrm{rep}_V\, e$ for all $e \in E$ is well-defined. *We state without proof that the composition*

$$(\overline{\mathrm{rep}}_V \circ \varepsilon) : U \otimes V \to \mathrm{Hom}_F(U^\dagger, V)$$

is actually a vector space homomorphism, and can thus be considered a *linear representation* of the tensor product space $U \otimes V$. Since rep_V is a *complete* invariant of E, the map $\overline{\mathrm{rep}}_V$ is injective. The composition $\overline{\mathrm{rep}}_V \circ \varepsilon$ is also injective, and can thus be considered a *faithful* linear representation of the tensor product space $U \otimes V$.

It follows that the tensor product space $U \otimes V$ is isomorphic to its image

$$(\overline{\mathrm{rep}}_V \circ \varepsilon)\,(U \otimes V) \subset \mathrm{Hom}_F(U^\dagger, V).$$

This image contains only finite-rank transformations in $\mathrm{Hom}_F(U^\dagger, V)$. Conversely, it is not difficult to show that every finite-rank transformation in $\mathrm{Hom}_F(U^\dagger, V)$ arises in this way.[8] We conclude that the space $U \otimes V$ is isomorphic to the subspace of finite-rank linear

[8] This converse is argued in [LC85, p. 3] using slightly different notation, with abstract Banach spaces taking the place of concrete function spaces over an arbitrary field, and topological dual spaces taking the place of abstract dual spaces; despite these differences, the argument is essentially the same.

transformations from a generalized dual space $U^\dagger$ into the vector space V. *We state without proof that this isomorphism is canonical—it can be defined abstractly, in a manner that is independent of the representations of tensor products by concrete expressions.*

Upon searching the mathematical literature electronically, the author was surprised to discover that no prior researchers had used the formal methods of computer algebra to develop algorithms, normal forms, and invariants for deciding the equivalence of tensor product expressions. A preliminary version of some of the author's results in this area appeared earlier in the author's doctoral thesis proposal [Cha00].

Among prior works with which the author is familiar, the work of Light and Cheney is the closest in spirit to the methods of computer algebra. In the previously cited works [LC85] and [Che86], Light and Cheney *define* the algebraic tensor product of two abstract Banach spaces to be the set of equivalence classes $E/\simeq$ of formal expressions in E with respect to various equivalence relations $\simeq$. Given two tensor product expressions $e_1, e_2 \in E$, Light and Cheney in [LC85, p. 1] *define* the equivalence relation $\simeq$ using the complete invariant rep_V (with the topological dual space U' in place of the generalized dual space $U^\dagger$) by asserting that $e_1 \simeq e_2$ *means* $\mathrm{rep}_V\, e_1 = \mathrm{rep}_V\, e_2$. Similarly, Cheney in [Che86, pp. 9–10] associates a *bilinear* transformation $\beta e : U' \times V' \to F$ with each tensor product expression

$$e := \sum_{i=0}^{n-1} u_i \otimes v_i$$

via

$$(\beta e)\,(\phi, \psi) := \sum_{i=0}^{n-1} \phi(u_i)\, \psi(v_i) \quad \text{for all} \quad (\phi, \psi) \in U' \times V',$$

and then *defines* the equivalence relation $\simeq$ by asserting that $e_1 \simeq e_2$ *means* $\beta e_1 = \beta e_2$. Cheney goes on to prove that $\beta e_1 = \beta e_2$ if and only if $\mathrm{rep}_V\, e_1 = \mathrm{rep}_V\, e_2$, which shows that the definition of $\simeq$ in [Che86] is equivalent to the definition of $\simeq$ given in [LC85].

In summary, the complete invariant rep_V provides the *foundation* of Light and Cheney's approach to algebraic tensor product spaces; in sharp contrast, rep_V is the *culmination* of this author's own approach to the same subject. Although the two approaches are mathematically equivalent, they do *not* involve an equivalent amount of work! This author prefers to closely follow the modern algebraic approach to tensor product theory for the following reason: As we saw at the beginning of this chapter, the universal property of algebraic tensor product spaces provide us with a *canonical* way to define linear transformations, from which it *automatically* follows that such transformations are well-defined on equivalence classes of expressions. This powerful technique simplifies both definitions and proofs alike. The universality of tensor products truly is a *labor-saving device!*

Chapter 5

Applications to Interpolation on Grid Lines

Thus far, we have built a partial foundation for the thesis from standard results in both abstract linear algebra and basic functional analysis, and we have fully explored the nonuniqueness of tensor product representations from an algorithmic point of view. Let us now take a scenic detour along the thesis highway, temporarily departing from the development of foundations in order to explore some interesting applications of the previous three chapters to the algebraic theory of multivariate interpolation.

We will begin by introducing an important special case of the asymptotic splitting operator—a case which can be described in a purely algebraic way. After we develop the fundamental properties of the asymptotic splitting operator in this special case, we will use the operator to develop an iterative algorithm which performs natural tensor product interpolation on the lines of a two-dimensional grid. We will conclude this chapter by revisiting the special case of the asymptotic splitting operator in order to abstract its structure, thereby hinting at the nature of the abstract splitting operator that is to come.

5.1 The Asymptotic Splitting Operator in a Special Case

This section introduces an important special case of the asymptotic splitting operator $\Upsilon_{(x_0,y_0)}$. Let $X, Y \subset \mathbb{R}$ be intervals and choose a point $(x_0, y_0) \in X \times Y$. Assume that $f : X \times Y \to \mathbb{R}$ is a continuous function. If $f(x_0, y_0) \neq 0$, we will later show that in this special case,

$$\Upsilon_{(x_0,y_0)} f(x,y) = \frac{f(x,y_0)\, f(x_0,y)}{f(x_0,y_0)} \quad \text{for all} \quad (x,y) \in X \times Y. \tag{5.1}$$

Note that $\Upsilon_{(x_0,y_0)} f$ is a tensor product of rank one, and that $\Upsilon_{(x_0,y_0)} f(x_0,y_0) \neq 0$.

Since equation (5.1) involves only algebraic operations, we can easily reformulate this result in a more general algebraic setting. We begin by replacing $\mathbb{R}$ by the arbitrary field F and introducing the following notation: If S is an arbitrary set and $\mathcal{F} \subset F^S$ is a function space, we define a special subset of $\mathcal{F}$ for fixed $s \in S$ by

$$\mathcal{F}_s := \{f \in \mathcal{F} : f(s) \neq 0\}.$$

Next, we let X and Y be arbitrary sets and choose a point $(x_0,y_0) \in X \times Y$. We can now define the operator

$$\Upsilon_{(x_0,y_0)} : \left(F^{X\times Y}\right)_{(x_0,y_0)} \to \left(F^X \otimes F^Y\right)_{(x_0,y_0)}$$

by equation (5.1) for all $f \in \left(F^{X\times Y}\right)_{(x_0,y_0)}$.

Note that the set $\left(F^{X\times Y}\right)_{(x_0,y_0)}$ is closed under pointwise multiplication and that the operator $\Upsilon_{(x_0,y_0)}$ is multiplicative. In addition, the fixed-points of $\Upsilon_{(x_0,y_0)}$ are precisely the rank-one tensor products in $\left(F^X \otimes F^Y\right)_{(x_0,y_0)}$. Consequently, the univariate functions in $\left(F^X\right)_{x_0}$ and in $\left(F^Y\right)_{y_0}$ are also fixed-points, as are the constant functions in $F\backslash\{0\}$. These properties together imply that the operator $\Upsilon_{(x_0,y_0)}$ is homogeneous for nonzero scalars.

Since the range of $\Upsilon_{(x_0,y_0)}$ is identical to the set of fixed-points of $\Upsilon_{(x_0,y_0)}$, the operator $\Upsilon_{(x_0,y_0)}$ is idempotent. Although $\Upsilon_{(x_0,y_0)}$ is homogeneous for nonzero scalars, $\Upsilon_{(x_0,y_0)}$ is *not linear* because it is *not additive*. We conclude that $\Upsilon_{(x_0,y_0)}$ has two of the three defining properties of a *projection*, which is an idempotent linear transformation on a vector space. Although projections play a central role in classical interpolation theory, we do not suffer from the lack of additivity in the operator $\Upsilon_{(x_0,y_0)}$. In this section, we will establish the very properties we will need to develop an *iterative interpolation scheme based on the operator* $\Upsilon_{(x_0,y_0)}$ in the next section.

To facilitate a discussion of the interpolation properties of $\Upsilon_{(x_0,y_0)}$, we now introduce some terminology inspired by geometry. We call the set $X \times Y$ a **rectangle**. In the special case $Y = X$, we call the set X^2 a **square**. If $x_0 \in X$, then the rectangle $X \times Y$ contains a subset $\{x_0\} \times Y$ which we call the **line** $x = x_0$. Similarly, if $y_0 \in Y$, then the subset

$X \times \{y_0\}$ is called the **line** $y = y_0$. *Note that the precise meaning of the lines* $x = x_0$ *and* $y = y_0$ *depends on the rectangle* $X \times Y$ *over which the lines are constructed!*

Given an arbitrary subset $S \subset X \times Y$ and two functions $f_1, f_2 \in F^{X \times Y}$, we say that f_1 **interpolates** f_2 **on** S if

$$f_1(x, y) = f_2(x, y) \quad \text{for all} \quad (x, y) \in S,$$

or more concisely, if the equality of restrictions $f_1|S = f_2|S$ holds. Given a *family* of arbitrary subsets $\{S_i\}_{i \in I} \subset 2^{X \times Y}$, we say that f_1 **interpolates** f_2 **on** $\{S_i\}_{i \in I}$ if f_1 interpolates f_2 on S_i *for each* $i \in I$, or equivalently, if f_1 interpolates f_2 on the *union* of subsets

$$\bigcup_{i \in I} S_i.$$

For example, to say that f_1 interpolates f_2 *on the lines* $x = x_0$ *and* $y = y_0$ means that f_1 interpolates f_2 *on the union of sets*

$$(\{x_0\} \times Y) \ \cup \ (X \times \{y_0\}) \, .$$

We are now ready to state the fundamental interpolation property of the asymptotic splitting operator:

Proposition 5.1 *If* $f \in \left(F^{X \times Y}\right)_{(x_0, y_0)}$, *then* $\Upsilon_{(x_0,y_0)} f$ *interpolates* f *on the lines* $x = x_0$ *and* $y = y_0$.

Proof. Our definitions, upon simplification, yield

$$\begin{aligned}
\Upsilon_{(x_0,y_0)} f(x_0, y) &= \frac{f(x_0, y_0)\, f(x_0, y)}{f(x_0, y_0)} = f(x_0, y) \quad \text{for all} \quad y \in Y, \\
\Upsilon_{(x_0,y_0)} f(x, y_0) &= \frac{f(x, y_0)\, f(x_0, y_0)}{f(x_0, y_0)} = f(x, y_0) \quad \text{for all} \quad x \in X,
\end{aligned}$$

which shows that $\Upsilon_{(x_0,y_0)} f$ has the desired interpolation properties. ■

This proposition says that the operator $\Upsilon_{(x_0,y_0)}$ performs interpolation on the two rectilinear coordinate lines $x = x_0$ and $y = y_0$ which pass through the point (x_0, y_0). Note that this is much stronger than performing interpolation merely at the single point (x_0, y_0). The geometry of these interpolation properties is illustrated in Figure 5.1.

If a function $r \in F^{X \times Y}$ interpolates 0 on the line $x = x_0$, we say that r **vanishes on the line** $x = x_0$, and we call $x = x_0$ a **zero line of** r. Similarly, if r interpolates 0 on

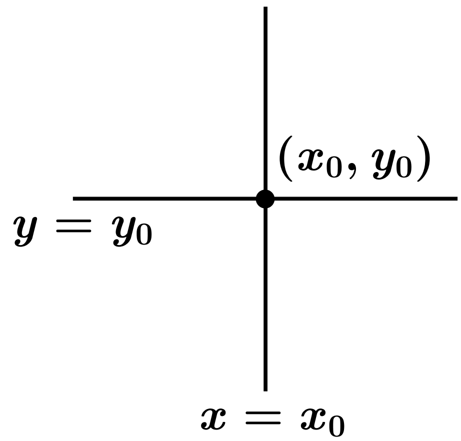

Figure 5.1: Interpolation on the Rectilinear Coordinate Lines $x = x_0$ and $y = y_0$

the line $y = y_0$, we say that r **vanishes on the line** $y = y_0$, and we call $y = y_0$ a **zero line of** r as well. Note that a zero line of r is a rectilinear coordinate line that consists entirely of zeros of r. The zero *lines* of a function of *two* real variables will play a role in the thesis analogous to the zero *points* (i.e., ordinary zeros) of a function of *one* real variable. The following corollary expresses the previous proposition in this new terminology, and provides a useful alternate point of view:

Corollary 5.2 *If $f \in \left(F^{X\times Y}\right)_{(x_0,y_0)}$, then the remainder $r := f - \Upsilon_{(x_0,y_0)} f$ vanishes on the lines $x = x_0$ and $y = y_0$.*

The following proposition establishes another fundamental property—*the asymptotic splitting operator preserves zero lines:*

Proposition 5.3 *Choose distinct $x_0, x_1 \in X$, distinct $y_0, y_1 \in Y$, and let $r \in \left(F^{X\times Y}\right)_{(x_1,y_1)}$ be arbitrary. If r vanishes on $x = x_0$, then $\Upsilon_{(x_1,y_1)}\, r$ vanishes on $x = x_0$ also. Similarly, if r vanishes on $y = y_0$, then $\Upsilon_{(x_1,y_1)}\, r$ vanishes on $y = y_0$ also.*

Proof. If r vanishes on $x = x_0$, then by definition,

$$\Upsilon_{(x_1,y_1)}\, r(x_0, y) = \frac{r(x_0, y_1)\, r(x_1, y)}{r(x_1, y_1)} = \frac{0 \cdot r(x_1, y)}{r(x_1, y_1)} = 0 \quad \text{for all} \quad y \in Y,$$

which means that $\Upsilon_{(x_1,y_1)}\, r$ vanishes on $x = x_0$ also. Similarly, if r vanishes on $y = y_0$, then by definition,

$$\Upsilon_{(x_1,y_1)}\, r(x, y_0) = \frac{r(x, y_1)\, r(x_1, y_0)}{r(x_1, y_1)} = \frac{r(x, y_1) \cdot 0}{r(x_1, y_1)} = 0 \quad \text{for all} \quad x \in X,$$

which means that $\Upsilon_{(x_1,y_1)}\, r$ vanishes on $y = y_0$ also. ■

Let us summarize what we have accomplished in this section while hinting at what awaits us in the next section: If $f \in \left(F^{X\times Y}\right)_{(x_0,y_0)}$, then the rank-one tensor product $s_1 := \Upsilon_{(x_0,y_0)} f$ interpolates f on the lines $x = x_0$ and $y = y_0$. This implies that the remainder $r_1 := f - s_1$ vanishes on the lines $x = x_0$ and $y = y_0$. If $r_1 \neq 0$ (in other words, if f is not a rank-one tensor product), we can select $(x_1, y_1) \in X \times Y$ with $r_1(x_1, y_1) \neq 0$. This implies that $x_1 \neq x_0$ and $y_1 \neq y_0$, and establishes that $r_1 \in \left(F^{X\times Y}\right)_{(x_1,y_1)}$.

Since the operator $\Upsilon_{(x_1,y_1)}$ preserves zero lines, the function $\Upsilon_{(x_1,y_1)}\, r_1$ vanishes on $x = x_0$ and $y = y_0$ just as r_1 does. As a result, the two-term tensor product $s_2 := s_1 + \Upsilon_{(x_1,y_1)}\, r_1$ *inherits the interpolation properties of* s_1: This means that s_2 interpolates f on the two *original* lines $x = x_0$ and $y = y_0$ just as s_1 does. In addition, since $\Upsilon_{(x_1,y_1)}\, r_1$ interpolates the remainder r_1 on $x = x_1$ and $y = y_1$, and since $s_1 + r_1 = f$ by definition, it follows that s_2 interpolates f on the two *new* lines $x = x_1$ and $y = y_1$ as well. In conclusion, we have constructed a two-term tensor product s_2 which interpolates f on *four* distinct lines $x = x_i$ and $y = y_i$ for $i = 0, 1$. In the next section, we will extend this process to construct an n-term tensor product s_n which interpolates f on the $2n$ distinct lines $x = x_i$ and $y = y_i$ for $i = 0, 1, \ldots, n-1$.

5.2 An Algorithm for Interpolation on Grid Lines

Let F denote an arbitrary field, let X and Y be arbitrary sets, and assume $\{(x_i, y_i)\}_{i=0}^{n-1} \subset X \times Y$. The previous section introduced a special case of the asymptotic splitting operator

$$\Upsilon_{(x_i,y_i)} : \left(F^{X\times Y}\right)_{(x_i,y_i)} \to \left(F^X \otimes F^Y\right)_{(x_i,y_i)}.$$

This section uses the operator $\Upsilon_{(x_i,y_i)}$ to iteratively construct an n-term tensor product $s_n \in F^X \otimes F^Y$ which interpolates the function $f \in F^{X\times Y}$ on the $2n$ distinct lines $x = x_i$ and $y = y_i$ for $i = 0, 1, \ldots, n-1$. To facilitate the discussion of this interpolation scheme, we now introduce some additional terminology inspired by geometry.

Intuitively, the simplest kind of rectangular $n \times n$ grid in $X \times Y$ depends on n parameters in X as well as n parameters in Y, and consists of $2n$ distinct lines which intersect in n^2 distinct points. We will formalize this intuition *by identifying a grid with its parameters and regarding a grid as a purely algebraic object*, namely a point in $X^n \times Y^n$. From this algebraic object, *we will then derive geometric structures*, such as the grid lines and their intersection points. Here are the formal definitions:

If $\Gamma_n \in X^n \times Y^n$, we call Γ_n a **rectangular grid of order** n **in** $X \times Y$. In particular, if

$$\Gamma_n = (x_0, x_1, \ldots, x_{n-1}; y_0, y_1, \ldots, y_{n-1}),$$

we call $x_0, x_1, \ldots, x_{n-1}$ the **parameters of** Γ_n **in** X and $y_0, y_1, \ldots, y_{n-1}$ the **parameters of** Γ_n **in** Y. If the parameters of Γ_n in X are distinct and the parameters of Γ_n in Y are distinct, we say that the grid Γ_n is **simple**.

Each of the lines $x = x_i$ and $y = y_i$ for $i = 0, 1, \ldots, n-1$ is called a **line of the grid** Γ_n. We formally define the collection of all such lines by

$$\text{lines of } \Gamma_n \;:=\; \{\{x_i\} \times Y\}_{i=0}^{n-1} \cup \{\{X \times \{y_i\}\}_{i=0}^{n-1}.$$

Note that the collected lines of Γ_n form a *family* of subsets of $X \times Y$. We call the *union* of all the subsets in this family the **points of the grid** Γ_n. More formally, we define

$$\text{points of } \Gamma_n \;:=\; \bigcup_{i=0}^{n-1} (\{x_i\} \times Y) \;\cup\; \bigcup_{i=0}^{n-1} (X \times \{y_i\}) \;=\; \left(\{x_i\}_{i=0}^{n-1} \times Y\right) \cup \left(X \times \{y_i\}_{i=0}^{n-1}\right).$$

Let us now define a special kind of grid point: We call each point of the form (x_i, y_j) for $i, j = 0, 1, \ldots, n-1$ an **intersection point of the grid** Γ_n. The intersection point (x_i, y_j) is so named because it is the point where the grid lines $x = x_i$ and $y = y_j$ intersect. We formally define the collection of all such points by

$$\text{intersection points of } \Gamma_n \;:=\; \{(x_i, y_j)\}_{i,j=0}^{n-1}.$$

Going further, we now define a special kind of intersection point: We call each point of the form (x_i, y_i) for $i = 0, 1, \ldots, n-1$ a **splitting point of the grid** Γ_n. The splitting point (x_i, y_i) is so named because its coordinates are the parameters of the asymptotic splitting operator $\Upsilon_{(x_i, y_i)}$. We formally define the collection of all such points by

$$\text{splitting points of } \Gamma_n \;:=\; \{(x_i, y_i)\}_{i=0}^{n-1}.$$

Coming full circle, we call the original Γ_n **the grid generated by the splitting points** $\{(x_i, y_i)\}_{i=0}^{n-1}$.

The following inclusions summarize the different kinds of points defined above:

$$\underbrace{\{(x_i, y_i)\}_{i=0}^{n-1}}_{\text{splitting points of } \Gamma_n} \subset \underbrace{\{(x_i, y_j)\}_{i,j=0}^{n-1}}_{\text{intersection points of } \Gamma_n} \subset \underbrace{\left(\{x_i\}_{i=0}^{n-1} \times Y\right) \cup \left(X \times \{y_i\}_{i=0}^{n-1}\right)}_{\text{points of } \Gamma_n} \subset X \times Y.$$

Figure 5.2 illustrates the parameters, lines, intersection points, and splitting points of a rectangular grid of order n; the splitting points are circled.

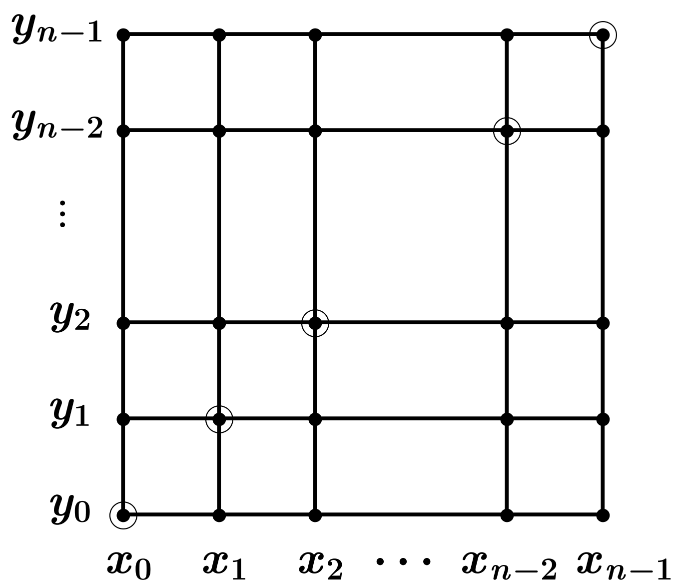

Figure 5.2: The Geometry of a Rectangular Grid of Order n

Note that by the definition given in the previous section, interpolation on the *lines* of the grid Γ_n (a *family* of subsets in $X \times Y$) is equivalent to interpolation on the *points* of the grid Γ_n (the *union* of the subsets in this family). We will use the former terminology to emphasize the strong nature of this kind of interpolation and avoid possible confusion between ordinary grid points and intersection points. Now that we have defined all the necessary terminology, we present the specification of an iterative algorithm which performs interpolation on grid lines using tensor products:

Algorithm 5.4 (Grid Interpolation) *Let the function $f \in F^{X \times Y}$ and integer $n \geq 1$ be such that either $f \notin F^X \otimes F^Y$, or $f \in F^X \otimes F^Y$ with* $\operatorname{rank} f \geq n$. *Now construct an n-term tensor product $s_n \in F^X \otimes F^Y$ and a rectangular grid $\Gamma_n \in X^n \times Y^n$ as follows:*

1. *Let $s_0 := 0$ and $r_0 := f$.*

2. *Iterate the following three steps for* $i := 0, 1, \ldots, n-1$*:*

   (a) *Choose a splitting point* $(x_i, y_i) \in X \times Y$ *such that* $r_i(x_i, y_i) \neq 0$.

   (b) *Calculate the next partial sum by* $s_{i+1} := s_i + \Upsilon_{(x_i,y_i)}\, r_i$.

   (c) *Calculate the next remainder by* $r_{i+1} := f - s_{i+1}$.

3. *Let* Γ_n *denote the grid generated by the splitting points* $\{(x_i, y_i)\}_{i=0}^{n-1}$.

4. *Return the tensor product* s_n *and grid* Γ_n.

If the input function f for this algorithm is not a tensor product, there is no upper bound on the number of iterations that we can carry out successfully. If f is a tensor product, then the number of iterations which can be successfully completed is bounded above rank n. These input requirements are reasonable since there would be little point in interpolating a tensor product f by another tensor product s_n with $n > \operatorname{rank} f$ terms—for this would mean that *the interpolant* s_n *has more than the minimal number of terms needed to represent the original tensor product* f *exactly.* We conclude that the input requirements of the algorithm cover all cases that are of genuine interest.

Note that Step 2a is the only step of the algorithm which could possibly fail. We would be unable to carry out this step only if $r_i = 0$ for some i, which would imply that $f = s_i$, where s_i is a tensor product with i terms. In that case, we would obtain

$$\operatorname{rank} f \leq i \leq n-1 < n,$$

which would contradict the input requirement that $\operatorname{rank} f \geq n$ whenever f is a tensor product. We conclude that this algorithm cannot fail if all the input requirements are satisfied. The following theorem establishes the interpolation properties resulting from the successful completion of this algorithm:

Theorem 5.5 (Grid Interpolation) *If the function* $f \in F^{X\times Y}$ *and integer* $n \geq 1$ *satisfy the input requirements of Grid Interpolation Algorithm 5.4, and the tensor product* $s_n \in F^X \otimes F^Y$ *and rectangular grid* $\Gamma_n \in X^n \times Y^n$ *denote the outputs of the algorithm, then the grid* Γ_n *is simple, and* s_n *interpolates* f *on the lines of the grid* Γ_n.

Proof. The proof is by finite induction on m. For $1 \leq m \leq n$, let P_m denote the proposition that after m iterations of Grid Interpolation Algorithm 5.4, the resulting grid Γ_m is simple and the resulting tensor product s_m interpolates f on the lines of the grid Γ_m. Proposition

P_1 certainly holds since the grid $\Gamma_1 := (x_0; y_0)$ is automatically simple and since $s_1 := \Upsilon_{(x_0,y_0)} f$ interpolates f on the lines $x = x_0$ and $y = y_0$ of the grid Γ_1 by Proposition 5.1.

Suppose that proposition P_m holds, where $1 \leq m \leq n-1$. We must show that proposition P_{m+1} holds also. By proposition P_m, the grid Γ_m is simple, which means the parameters $x_0, x_1, \ldots, x_{m-1}$ are distinct and the parameters $y_0, y_1, \ldots, y_{m-1}$ are distinct. Suppose that $x_m = x_i$ for some $i \in \{0, 1, \ldots, m-1\}$. Since $x = x_i$ is a line of the grid Γ_m, the tensor product s_m interpolates f on the line $x = x_i$ by proposition P_m, and the remainder $r_m := f - s_m$ therefore vanishes on the line $x = x_i$. This implies that $r_m(x_m, y_m) = r_m(x_i, y_m) = 0$, which contradicts that $r_m(x_m, y_m) \neq 0$. We conclude that the parameters $x_0, x_1, \ldots, x_m$ must be distinct. By a similar argument, the parameters $y_0, y_1, \ldots, y_m$ are also distinct, and thus the grid Γ_{m+1} is simple.

To show that s_{m+1} interpolates f on the lines of the grid Γ_{m+1}, we will first consider the lines of the grid Γ_m and then consider the lines $x = x_m$ and $y = y_m$ afterwards. Since s_m interpolates f on the lines of Γ_m by proposition P_m, the remainder $r_m := f - s_m$ vanishes on the lines of Γ_m. Since the grid Γ_{m+1} is simple and $r_m(x_m, y_m) \neq 0$, the hypotheses of Proposition 5.3 are satisfied, and thus $\Upsilon_{(x_m,y_m)} r_m$ also vanishes on the lines of Γ_m. As a result, $s_{m+1} := s_m + \Upsilon_{(x_m,y_m)} r_m$ interpolates s_m on the lines of Γ_m, and s_m in turn interpolates f on the lines of Γ_m; hence, s_{m+1} interpolates f on the lines of Γ_m. Since $\Upsilon_{(x_m,y_m)} r_m$ interpolates r_m on the lines $x = x_m$ and $y = y_m$ by Proposition 5.1, $s_{m+1} := s_m + \Upsilon_{(x_m,y_m)} r_m$ interpolates $f = s_m + r_m$ on the lines $x = x_m$ and $y = y_m$. We conclude that s_{m+1} interpolates f on the lines of $\Gamma_{m+1.}$

Since proposition P_1 holds, and proposition P_m implies proposition P_{m+1} whenever $1 \leq m \leq n-1$, we conclude that proposition P_m holds for all m satisfying $1 \leq m \leq n$. In particular, proposition P_n holds, as required. ■

Here are some interesting facts, *which we will state without proof:* The tensor product output s_n of Grid Interpolation Algorithm 5.4 actually has *rank* n, not just n *terms.* If the input f is *already* a tensor product of rank n, the algorithm produces an exact result after n iterations—meaning that $r_n = 0$ and $s_n = f$. This, in turn, gives us a sufficient condition for proving that a function f is *not* a tensor product: find an infinite sequence of splitting points $\{(x_i, y_i)\}_{i=0}^{\infty}$ such that $r_i(x_i, y_i) \neq 0$ for all $i \geq 0$. For example, the author has used this criterion to show that the piecewise linear function $f(x, y) = |x - y|$ is *not* a tensor product—a fact which seems obvious, but whose proof is certainly not trivial!

5.3 The Asymptotic Splitting Operator Revisited

We will now end this chapter by revisiting the special case of the asymptotic splitting operator developed at the beginning of the chapter. Let X and Y be arbitrary sets, and let F be an arbitrary field. Choose a function $f : X \times Y \to F$ and a point $(x_0, y_0) \in X \times Y$. Recall that when $f(x_0, y_0) \neq 0$, we defined

$$\Upsilon_{(x_0,y_0)} f(x, y) := \frac{f(x, y_0)\, f(x_0, y)}{f(x_0, y_0)} \quad \text{for all} \quad (x, y) \in X \times Y.$$

The numerator of this definition can be rewritten in terms of evaluation operators and expressed as a tensor product:

$$f(x, y_0)\, f(x_0, y) = (E^{y_0} f)(x)\, (E_{x_0} f)(y) =: (E^{y_0} f \otimes E_{x_0} f)(x, y).$$

The denominator can be rewritten in terms of the linear functional generated by composing the evaluation operators:

$$f(x_0, y_0) = E_{x_0} E^{y_0} f.$$

Together, these transformations yield the reformulation

$$\Upsilon_{(x_0,y_0)} f(x, y) = \frac{(E^{y_0} f \otimes E_{x_0} f)\,(x, y)}{E_{x_0} E^{y_0} f} \quad \text{for all} \quad (x, y) \in X \times Y.$$

Eliminating the independent variables allows us to write this more simply in function notation as

$$\Upsilon_{(x_0,y_0)} f = \frac{E^{y_0} f \otimes E_{x_0} f}{E_{x_0} E^{y_0} f}.$$

In order to simplify the notation further, let $\Phi := E_{x_0}$ and $\Psi := E^{y_0}$ denote the parametric extensions of the linear functionals $\phi := \varepsilon_{x_0}$ and $\psi := \varepsilon^{y_0}$; these choices conform to the general notational conventions we adopted in Section 3.2. We can now rewrite the *asymptotic* splitting operator $\Upsilon_{(x_0,y_0)}$ at the point (x_0, y_0) as an *abstract* splitting operator $\Omega_{(\phi,\psi)}$ induced by the pair of linear functionals (ϕ, ψ); this final transformation yields

$$\Omega_{(\phi,\psi)} f = \frac{\Psi f \otimes \Phi f}{\Phi \Psi f}.$$

The reformulation of $\Upsilon_{(x_0,y_0)}$ as $\Omega_{(\phi,\psi)}$ in this special case gives us a sneak preview of the author's abstract splitting operator. This reformulation also reveals that the class of series expansions developed in this chapter—series which perform natural tensor product interpolation on the lines of a two-dimensional grid—are examples of the more general class of series expansions which the author calls Geddes series.

Chapter 6

Foundations from Asymptotic Analysis

Now that we have refreshed ourselves with an enjoyable scenic detour, we are ready to continue with the painstaking work of building a suitable mathematical foundation for this thesis. In this chapter, we will draw upon some of the basic topological notions of Chapter 3 in order to study the asymptotic behavior of a real-valued function of one real variable near a limit point of its domain. We will develop this univariate theory using largely standard asymptotic methods such as those described in [Erd56], [dB70], and [BH75].

We will also build on this standard material in an original way—and to a considerable degree—thereby completing the preliminary work for a completely new foundation for asymptotic analysis in one real variable. The author's motivation for developing a new foundation for this time-honored subject is twofold:

1. The author is simply not satisfied with the level of rigor generally employed in classical asymptotic analysis. Prior work in this area routinely omits fundamental hypotheses, and often fails to clearly articulate vital assumptions. The author believes that the painstakingly careful contributions of this chapter will go a long way towards developing a satisfactory solution to this ongoing problem. Moreover, when we pay extremely careful attention to hypotheses, we are amply rewarded by the discovery of whole new mathematical worlds whose existence was previously unknown to us; such epiphanies lead to advances in theory which in turn make it possible to pursue new applications.

2. The author has come to realize that the fundamentals of asymptotic analysis, aspects of interpolation theory, and the fundamentals of classical inner product space theory

can all be absorbed into a much more general theoretical framework which unifies these disparate branches of mathematics. Although a complete exposition of this program of abstraction and unification unfortunately lies outside of the scope of the thesis, let us seize this opportunity to advance that program as far as possible as it pertains to asymptotic analysis.

We begin this chapter with a traditional look at the traditional asymptotic order relations that provide the customary foundation for classical asymptotic analysis. We will follow this immediately with a much deeper, nontraditional look at the most important traditional asymptotic order relation, and from this closer examination will articulate the classes of functions that provide suitable hypothesis for rigorous work in asymptotic analysis. After this, we will develop an original asymptotic order relation which plays an important role in both old and new foundations of asymptotic analysis. We will conclude this chapter by developing the basic facts about asymptotic sequences, asymptotic series, and asymptotic expansions from both the traditional and nontraditional points of view.

6.1 Standard Asymptotic Order Relations

This section begins our study of asymptotic analysis by defining the Landau O and o order relations which are the traditional foundation of the subject. We will define the Θ order relation as an additional convenience. Later in the thesis, we will find the O and Θ relations particularly useful in the remainder and convergence theories for the interpolation and approximation of functions of one or two real variables.

Let X be an arbitrary set, let $f, g : X \to \mathbb{R}$ be arbitrary functions, and let $S \subset X$ be an arbitrary subset. We say that $f(x)$ **is "big oh" of** $g(x)$ **for all** $x \in S$ if there exists a real constant $M > 0$ such that

$$|f(x)| \leq M\,|g(x)| \quad \text{for all} \quad x \in S.$$

We denote this relation by

$$f(x) = O(g(x)) \quad \text{for all} \quad x \in S.$$

If $g(x) \neq 0$ for all $x \in S$, this relation says that the quotient f/g is bounded on S.

Although O defines a *relation* on the set $\mathbb{R}^X$, this relation is *not an equivalence relation*—for although O is reflexive and transitive, O *is not symmetric*. For example,

$x^2 = O(x)$ for all $x \in [-1, 1]$, but $x \neq O(x^2)$ for all $x \in [-1, 1]$. To compensate for this lack of symmetry, we define a new order relation Θ by *symmetrizing* the original order relation O as follows: We say that $f(x)$ **is "theta" of** $g(x)$ **for all** $x \in S$ if

$$f(x) = O(g(x)) \quad \text{and} \quad g(x) = O(f(x)) \quad \text{for all} \quad x \in S.$$

We denote this relation by

$$f(x) = \Theta(g(x)) \quad \text{for all} \quad x \in S.$$

The relation Θ is symmetric by definition, and inherits the reflexivity and transitivity of the relation O; hence, Θ is an equivalence relation on the set $\mathbb{R}^X$.

Written in terms of inequalities, the equivalence relation $f(x) = \Theta(g(x))$ for all $x \in S$ means that there exist real constants $M, N > 0$ such that

$$M\,|f(x)| \leq |g(x)| \leq N\,|f(x)| \quad \text{for all} \quad x \in S.$$

This equivalence relation ensures that *the magnitudes of the functions f and g vary together*—when one approaches zero, the other approaches zero, and when one approaches infinity, the other approaches infinity. Here is a familiar application of this equivalence relation to normed linear spaces: If X is a linear space over $\mathbb{R}$, and f and g are norms on X, the relation $f(x) = \Theta(g(x))$ for all $x \in X$ means that f and g are *equivalent norms*.

In the topology of the extended reals $\overline{\mathbb{R}}$, we define an **open neighborhood of** $-\infty$ to be an interval of the form $[-\infty, r)$ for any $r \in \mathbb{R}$. Similarly, we define an **open neighborhood of** ∞ to be an interval of the form $(r, \infty]$ for any $r \in \mathbb{R}$. We define an **open neighborhood of** $x_0 \in \mathbb{R}$ to be an interval of the form $(x_0 - \delta, x_0 + \delta)$ for any real $\delta > 0$. It is worth noting that if N_1 and N_2 are any two open neighborhoods of an arbitrary point $x_0 \in \overline{\mathbb{R}}$, then the intersection $N_1 \cap N_2$ is also an open neighborhood of x_0.

Assume that $X \subset \overline{\mathbb{R}}$, and let x_0 be a limit point of X. Note that x_0 may or may not belong to X. Given arbitrary functions $f, g : X \to \mathbb{R}$, we say that $f(x)$ **is "big oh" of** $g(x)$ **as** x **approaches** x_0 if there exists an open neighborhood N of the point x_0 such that $f(x) = O(g(x))$ for all $x \in (X \cap N)\backslash\{x_0\}$. We denote this relation by

$$f(x) = O(g(x)) \quad \text{as} \quad x \to x_0.$$

If $g(x) \neq 0$ for all $x \in (X \cap N)\backslash\{x_0\}$, this relation says that the quotient $f(x)/g(x)$ remains bounded as $x \to x_0$.

The definition of the O relation in both [Erd56, p. 5] and [BH75, p. 6, Dover] replaces the set $(X \cap N)\backslash\{x_0\}$ with the superset $X \cap N$, which imposes a slightly stronger condition that strengthens the definition of the O relation. In *typical* applications of asymptotics, the limit point x_0 does *not* belong to X, in which case the sets $(X \cap N)\backslash\{x_0\}$ and $X \cap N$ are the same, and this distinction is unimportant; however, since we are *also* interested in applications of asymptotics to the interpolation of functions, we must take special care to properly handle the case in which x_0 *does* belong to X.

If $x_0 \in X$, then $X \cap N$ is an open neighborhood of x_0 *in the subspace topology of* X, and $(X \cap N)\backslash\{x_0\}$ is a *deleted neighborhood of* x_0. It is universal practice in general topology, metric space theory, and elementary calculus to use *deleted neighborhoods of* x_0 when discussing the limiting behavior of a function as $x \to x_0$. In keeping with this established practice, the author has chosen to define *all* order relations as $x \to x_0$ using *deleted neighborhoods of* x_0. This choice facilitates the rigorous reformulation of these order relations in terms of limits. In the case where this distinction matters, namely $x_0 \in X$, nothing essential is lost by weakening the definition of these order relations in this way—for we can often invoke additional hypotheses, such as the continuity of f and g at x_0, to extend these order relations to hold at x_0.

Continuing in this fashion, we say that $f(x)$ **is "theta" of** $g(x)$ **as** x **approaches** x_0 if there exists an open neighborhood N of the point x_0 such that $f(x) = \Theta(g(x))$ for all $x \in (X \cap N)\backslash\{x_0\}$. We denote this equivalence relation by

$$f(x) = \Theta(g(x)) \quad \text{as} \quad x \to x_0.$$

It follows immediately that $f(x) = \Theta(g(x))$ as $x \to x_0$ if and only if $f(x) = O(g(x))$ and $g(x) = O(f(x))$ both hold as $x \to x_0$.

These general definitions accommodate all the typical applications of interest in asymptotics. For example, if $X = \mathbb{R}$, then ∞ is a limit point of $\mathbb{R}$, and an open neighborhood of ∞ is an interval N of the form $(r, \infty]$ for any $r \in \mathbb{R}$. The intersection $X \cap N$ has the form (r, ∞), which already excludes the limit point ∞. The relation $f(x) = O(g(x))$ as $x \to \infty$ therefore means that there exists a real number r such that $f(x) = O(g(x))$ for all real $x > r$. Similarly, 0 is a limit point of $\mathbb{R}$, and the relation $f(x) = O(g(x))$ as $x \to 0$ means that there exists a real number $\delta > 0$ such that $f(x) = O(g(x))$ for all real x with $0 < |x| < \delta$.

Note in particular that if $X = \mathbb{N}$, then ∞ is a limit point of $\mathbb{N}$ in the topology of $\overline{\mathbb{R}}$.

The relation $f(x) = O(g(x))$ as $x \to \infty$ means that there exists a natural number N such that $f(n) = O(g(n))$ for all natural numbers $n \geq N$. In this case, we regard the functions $f, g : \mathbb{N} \to \mathbb{R}$ as sequences $\{f_n\}_{n=0}^{\infty}, \; \{g_n\}_{n=0}^{\infty} \subset \mathbb{R}$, letting $f_n := f(n)$ and $g_n := g(n)$, and we write the O relation as

$$f_n = O(g_n) \quad \text{as} \quad n \to \infty.$$

Similarly, the equivalence relation $f(x) = \Theta(g(x))$ as $x \to \infty$ means that there exists a natural number N such that $f(n) = \Theta(g(n))$ for all natural numbers $n \geq N$, and we write the Θ equivalence relation in this case as

$$f_n = \Theta(g_n) \quad \text{as} \quad n \to \infty.$$

Assume once again that $X \subset \overline{\mathbb{R}}$, and let x_0 be a limit point of X. Given arbitrary functions $f, g : X \to \mathbb{R}$, we say that $f(x)$ **is "little oh" of** $g(x)$ **as** x **approaches** x_0 if for every $\epsilon > 0$, there exists an open neighborhood N_ϵ of the point x_0 such that

$$|f(x)| \leq \epsilon \, |g(x)| \quad \text{for all} \quad x \in (X \cap N_\epsilon) \backslash \{x_0\}.$$

We denote this relation by

$$f(x) = o(g(x)) \quad \text{as} \quad x \to x_0.$$

If there is some fixed open neighborhood N of x_0 such that $g(x) \neq 0$ for all $x \in (X \cap N) \backslash \{x_0\}$, then this relation says that the quotient $f(x)/g(x)$ approaches zero as $x \to x_0$. In applications of the o relation to asymptotic expansions, it is almost always the case that g does not vanish on some deleted neighborhood of x_0; in fact, we generally use this condition to calculate the coefficients of such expansions. The author therefore proposes the following terminology and notation to formalize this *de facto* hypothesis:

Definition 6.1 *Let $X \subset \overline{\mathbb{R}}$, let x_0 be a limit point of X, and let $g : X \to \mathbb{R}$ be an arbitrary function. If there exists an open neighborhood N of x_0 such that $g(x) \neq 0$ for all $x \in (X \cap N) \backslash \{x_0\}$, we say that g* ***is locally nonvanishing near*** *x_0. We denote the set of all such real-valued functions on X by $\mathrm{LNV}_{x_0}(X)$.*

The hypothesis $g \in \mathrm{LNV}_{x_0}(X)$ ensures that the quotient f/g is defined near x_0. This allows us to reformulate the relation $f(x) = o(g(x))$ as $x \to x_0$ in terms of a limit as follows:

Proposition 6.2 *Let $X \subset \overline{\mathbb{R}}$ and let x_0 be a limit point of X. If $f : X \to \mathbb{R}$ is an arbitrary function and $g \in \mathrm{LNV}_{x_0}(X)$, then the relation*

$$f(x) = o(g(x)) \quad \textit{as} \quad x \to x_0$$

holds if and only if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

This proposition follows directly from the definitions of the constructs involved. *Note that we must interpret the limit as $x \to x_0$ in a suitable manner when constrained by the geometry of X.* For example, if $X \subset \overline{\mathbb{R}}$ is an interval and x_0 is an endpoint of X, then the limit as $x \to x_0$ must be interpreted as a one-sided limit. If $X = \mathbb{N}$ and $x_0 = \infty$, then the limit as $x \to x_0$ must be interpreted as a limit over the natural numbers.

This concludes our brief introduction to the O, Θ, and o order relations. Since we will later define asymptotic sequences, asymptotic series, and asymptotic expansions using the o relation, we will study this particular relation in greater depth in the next section.

6.2 A Deeper Look at Landau's "Little Oh" Relation

Let $X \subset \overline{\mathbb{R}}$, let x_0 be a limit point of X, and let $f : X \to \mathbb{R}$ be an arbitrary function. Although the hypothesis $g \in \mathrm{LNV}_{x_0}(X)$ and Proposition 6.2 will suffice in most practical applications of asymptotics, it is useful for the sake of the theory to have more general criteria that determine precisely when the relation $f(x) = o(g(x))$ as $x \to x_0$ holds for an arbitrary function $g : X \to \mathbb{R}$. The key to developing such criteria is to consider the set $X \backslash g^{-1}(0)$ of points on which g does *not* vanish; this yields two distinct cases depending on whether x_0 is a limit point of this set.

We call the set $X \backslash g^{-1}(0)$ the **algebraic support of** g, and we call the *closure* of this set (*in the subspace topology of X*, not in the topology of $\overline{\mathbb{R}}$) the **topological support of** g. We denote the algebraic and topological supports of g by $\operatorname{a-supp} g$ and $\operatorname{t-supp} g$ respectively. In the mathematical literature, the topological support of g is called simply the support of g and denoted by $\operatorname{supp} g$; however, since g can vanish at a point in its topological support, the algebraic support better serves our immediate needs.

If x_0 is *not* a limit point of $\operatorname{a-supp} g$, then some deleted neighborhood of x_0 has empty intersection with $\operatorname{a-supp} g$, which means that g vanishes on that deleted neighborhood. The author proposes the following terminology and notation to formalize this property:

Definition 6.3 *Let $X \subset \overline{\mathbb{R}}$, let x_0 be a limit point of X, and let $g : X \to \mathbb{R}$ be an arbitrary function. If there exists an open neighborhood N of x_0 such that $g(x) = 0$ for all $x \in (X \cap N) \backslash \{x_0\}$, we say that g* ***is locally vanishing near*** *x_0. We denote the set of all such real-valued functions on X by $\mathrm{LV}_{x_0}(X)$.*

Under the hypothesis $g \in \mathrm{LV}_{x_0}(X)$, we obtain the following trivial criterion for determining when the relation $f(x) = o(g(x))$ as $x \to x_0$ holds:

Proposition 6.4 *Let $X \subset \overline{\mathbb{R}}$ and let x_0 be a limit point of X. If $f : X \to \mathbb{R}$ is an arbitrary function and $g \in \mathrm{LV}_{x_0}(X)$, then the relation*

$$f(x) = o(g(x)) \quad \textit{as} \quad x \to x_0$$

holds if and only if $f \in \mathrm{LV}_{x_0}(X)$.

In general, the relation $f(x) = o(g(x))$ as $x \to x_0$ implies that every zero of g which is sufficiently close to x_0 is also a zero of f. Equivalently, every point of the algebraic support of f which is sufficiently close to x_0 also lies in the algebraic support of g. The author proposes the following formalization of this local property:

Definition 6.5 *Let $x_0 \in \overline{\mathbb{R}}$ be an arbitrary point and let $S, T \subset \overline{\mathbb{R}}$ be arbitrary subsets. If there exists an open neighborhood N of x_0 such that*

$$(S \cap N) \backslash \{x_0\} \subset (T \cap N) \backslash \{x_0\},$$

we say that S ***is locally contained in*** *T* ***near*** *x_0, and we denote this property by*

$$S \subset_{x_0} T.$$

Note that $\subset_{x_0}$ is a reflexive and transitive relation on the power set $2^{\overline{\mathbb{R}}}$. It is not an equivalence relation because it is not symmetric; for example, $\varnothing \subset_{x_0} \overline{\mathbb{R}}$, but $\overline{\mathbb{R}} \not\subset_{x_0} \varnothing$. Just as we symmetrized the O relation to obtain the Θ *equivalence relation* on $\mathbb{R}^X$, we can symmetrize the $\subset_{x_0}$ relation to obtain a new $=_{x_0}$ *equivalence relation* on $2^{\overline{\mathbb{R}}}$: We say that S **is locally equal to** T **near** x_0 if $S \subset_{x_0} T$ and $T \subset_{x_0} S$ both hold, and we denote this equivalence relation by $S =_{x_0} T$. It follows immediately that $S =_{x_0} T$ if and only if there exists an open neighborhood N of x_0 such that

$$(S \cap N) \backslash \{x_0\} = (T \cap N) \backslash \{x_0\}.$$

In this new notation, the relation $f(x) = o(g(x))$ as $x \to x_0$ implies

$$\text{a-supp}\, f \subset_{x_0} \text{a-supp}\, g.$$

This condition is necessary for the order relation to hold, but not sufficient; however, we can augment this condition with another property to obtain a sufficient condition in the case that x_0 is a limit point of a-supp g. The author proposes the following terminology and notation to formalize the second of the two cases under consideration:

Definition 6.6 *Let $X \subset \overline{\mathbb{R}}$, let x_0 be a limit point of X, and let $g : X \to \mathbb{R}$ be an arbitrary function. If x_0 is a limit point of* a-supp g, *we say that g* ***is sequentially nonvanishing near*** x_0. *We denote the set of all such real-valued functions on X by $\mathrm{SNV}_{x_0}(X)$. By the definition of a limit point, $g \in \mathrm{SNV}_{x_0}(X)$ if and only if there exists a sequence*

$$\{\xi_n\}_{n=0}^{\infty} \subset (\text{a-supp}\, g)\backslash\{x_0\}$$

such that $\xi_n \to x_0$ as $n \to \infty$. We call such a sequence a ***supporting sequence for*** g ***near*** x_0.

A supporting sequence for g near x_0 is a sequence which avoids x_0 itself but converges to x_0 *while passing through the algebraic support of* g. Since we will have many occasions to consider a limit that is taken over all supporting sequences of a given function, the author proposes the following convenient terminology and notation:

Definition 6.7 *Let $X \subset \overline{\mathbb{R}}$, let x_0 be a limit point of X, and assume that $g \in \mathrm{SNV}_{x_0}(X)$. Let $h : X \to \mathbb{R}$ be an arbitrary function. If there exists a real number L such that*

$$\lim_{n\to\infty} h(\xi_n) = L$$

for every supporting sequence $\{\xi_n\}_{n=0}^{\infty}$ for g near x_0, we say that the ***limit of*** $h(x)$ ***as*** x ***approaches*** x_0 ***sequentially (through the algebraic support of*** g***) exists and equals*** L, *and we denote this property by*

$$\lim_{\substack{x \rightsquigarrow x_0 \\ g}} h(x) = L.$$

We call this kind of limit a ***supporting sequential limit (with respect to*** g***).***

The supporting sequential limit inherits all of the useful properties of the ordinary limit: The value of the supporting sequential limit is unique whenever it exists, and the supporting sequential limit is both linear and multiplicative. The following proposition tells us that a supporting sequential limit with respect to $g \in \mathrm{SNV}_{x_0}(X)$ reduces to an ordinary limit in the special case $g \in \mathrm{LNV}_{x_0}(X)$:

Proposition 6.8 *Let $X \subset \overline{\mathbb{R}}$, let x_0 be a limit point of X, and assume that $g \in \mathrm{LNV}_{x_0}(X)$. Let $h : X \to \mathbb{R}$ be an arbitrary function. The ordinary limit of $h(x)$ as $x \to x_0$ exists if and only if the supporting sequential limit of $h(x)$ as $x \underset{g}{\rightsquigarrow} x_0$ exists. If both limits exist, then their values are equal, and we write*

$$\lim_{x \to x_0} h(x) = \lim_{x \underset{g}{\rightsquigarrow} x_0} h(x).$$

With these new constructs at our disposal, we are now fully prepared to analyze the second of the two cases under consideration. Assume that $g \in \mathrm{SNV}_{x_0}(X)$. For convenience, let $X_0 :=$ a-supp g, and define the restrictions $F := f|X_0$ and $G := g|X_0$. By hypothesis, x_0 is a limit point of X_0, which is the domain of the functions F and G; consequently, we can consider whether the relation $F(x) = o(G(x))$ as $x \to x_0$ holds.

We find that the relation $f(x) = o(g(x))$ as $x \to x_0$ holds if and only if the local containment a-supp $f \subset_{x_0}$ a-supp g holds and the relation $F(x) = o(G(x))$ as $x \to x_0$ holds. Why is this? The relation $f(x) = o(g(x))$ as $x \to x_0$ clearly implies that a-supp $f \subset_{x_0}$ a-supp g and $F(x) = o(G(x))$ as $x \to x_0$ both hold. Conversely, the local containment a-supp $f \subset_{x_0}$ a-supp g is equivalent to $g^{-1}(0) \subset_{x_0} f^{-1}(0)$ and ensures that $f(x) = o(g(x))$ holds for all x near x_0 for which $g(x) = 0$, while the relation $F(x) = o(G(x))$ as $x \to x_0$ ensures that $f(x) = o(g(x))$ holds for all x near x_0 for which $g(x) \neq 0$.

Since $G(x) \neq 0$ for all $x \in X_0$, it follows that $G \in \mathrm{LNV}_{x_0}(X_0)$. Proposition 6.2 further implies that the relation $F(x) = o(G(x))$ as $x \to x_0$ holds if and only if

$$\lim_{x \to x_0} \frac{F(x)}{G(x)} = 0.$$

By Proposition 6.8, the above limit holds if and only if

$$\lim_{x \underset{G}{\rightsquigarrow} x_0} \frac{F(x)}{G(x)} = 0.$$

Since the functions G and g have the same algebraic support, G and g have the same supporting sequences near x_0, which means that

$$\lim_{\substack{x \rightsquigarrow x_0 \\ G}} \frac{F(x)}{G(x)} = \lim_{\substack{x \rightsquigarrow x_0 \\ g}} \frac{f(x)}{g(x)}.$$

We conclude that the relation $F(x) = o(G(x))$ as $x \to x_0$ holds if and only if

$$\lim_{\substack{x \rightsquigarrow x_0 \\ g}} \frac{f(x)}{g(x)} = 0.$$

Our analysis culminates in the following criterion for determining when the relation $f(x) = o(g(x))$ as $x \to x_0$ holds under the hypothesis $g \in \mathrm{SNV}_{x_0}(X)$:

Proposition 6.9 *Let $X \subset \overline{\mathbb{R}}$ and let x_0 be a limit point of X. If $f : X \to \mathbb{R}$ is an arbitrary function and $g \in \mathrm{SNV}_{x_0}(X)$, then the relation*

$$f(x) = o(g(x)) \quad \textit{as} \quad x \to x_0$$

holds if and only

$$\text{a-supp}\, f \subset_{x_0} \text{a-supp}\, g \quad \textit{and} \quad \lim_{\substack{x \rightsquigarrow x_0 \\ g}} \frac{f(x)}{g(x)} = 0.$$

In summary, Propositions 6.2, 6.4, and 6.9 give us three criteria that we can apply to determine when the relation $f(x) = o(g(x))$ as $x \to x_0$ holds. The particular criterion that we apply depends on whether g is locally nonvanishing, locally vanishing, or sequentially nonvanishing near x_0. These three potential properties of the function g yield a comprehensive classification scheme for the elements of $\mathbb{R}^X$, which we now summarize:

Considering whether x_0 is a limit point of a-supp g yields exactly two mutually exclusive cases; therefore, we can write the function space $\mathbb{R}^X$ as a disjoint union in the following way:

$$\mathbb{R}^X = \mathrm{LV}_{x_0}(X) \cup \mathrm{SNV}_{x_0}(X) \quad \text{where} \quad \mathrm{LV}_{x_0}(X) \cap \mathrm{SNV}_{x_0}(X) = \varnothing.$$

In addition, since every function which is *locally* nonvanishing near x_0 is also *sequentially* nonvanishing near x_0, the following inclusion holds:

$$\mathrm{LNV}_{x_0}(X) \subset \mathrm{SNV}_{x_0}(X).$$

Notice that neither $\mathrm{LNV}_{x_0}(X)$ nor $\mathrm{SNV}_{x_0}(X)$ is a subspace of the function space $\mathbb{R}^X$. The reason is that a subspace would have to contain the constant function zero, which is locally vanishing near x_0. In contrast, $\mathrm{LV}_{x_0}(X)$ is a subspace of the function space $\mathbb{R}^X$.

Note that the quotient space $\mathbb{R}^X / \mathrm{LV}_{x_0}(X)$ is a very natural setting for performing asymptotic analysis as $x \to x_0$. The nonzero elements of this quotient space consists of equivalence classes of functions which are sequentially nonvanishing near x_0, where the equivalence relation consists of the equality of functions on a deleted neighborhood of x_0. These nonzero equivalence classes describe the most general functions that are useful in asymptotic analysis. We choose not to pursue the quotient space formalism here, however, since we wish to avoid further complications; in what follows, we will focus instead on the individual elements of the class of functions $\mathrm{SNV}_{x_0}(X)$.

The following two examples contrast the virtues of the special class $\mathrm{LNV}_{x_0}(X)$ and the general class $\mathrm{SNV}_{x_0}(X)$. Let $X := \mathbb{R}\backslash\{0\}$ and note that $x_0 := 0$ is a limit point of X. Since the polynomial x is locally nonvanishing near 0, Proposition 6.2 tells us that the relation $x^2 = o(x)$ as $x \to 0$ holds because

$$\lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0.$$

In contrast, the transcendental expression $x \sin(1/x)$ fails to be *locally* nonvanishing near 0. The reason is that every deleted neighborhood of 0 contains a point of the form $z_n := 1/(n\pi)$ for some positive integer n, and $z_n \sin(1/z_n) = z_n \sin(n\pi) = 0$ for all $n \geq 1$. Now consider the sequence $\{\xi_n\}_{n=0}^{\infty} \subset \mathbb{R}\backslash\{0\}$ defined by $\xi_n := \left[(n+\frac{1}{2})\pi\right]^{-1}$ for all $n \in \mathbb{N}$. Since $\xi_n \to 0$ as $n \to \infty$ and

$$\xi_n \sin\frac{1}{\xi_n} = \xi_n \cdot (-1)^n \neq 0 \quad \text{for all} \quad n \in \mathbb{N},$$

$\{\xi_n\}_{n=0}^{\infty}$ is a supporting sequence for $x\sin(1/x)$ near 0. This implies that $x\sin(1/x)$ is *sequentially* nonvanishing near 0. Proposition 6.9 tells us that the relation

$$\left(x \sin\frac{1}{x}\right)^2 = o\left(x \sin\frac{1}{x}\right) \quad \text{as} \quad x \to 0$$

holds because $(x\sin(1/x))^2 \neq 0$ implies that $x\sin(1/x) \neq 0$, and

$$\lim_{n\to\infty} \frac{\left(\xi_n \sin\dfrac{1}{\xi_n}\right)^2}{\xi_n \sin\dfrac{1}{\xi_n}} = \lim_{n\to\infty} \xi_n \sin\frac{1}{\xi_n} = 0$$

for every supporting sequence $\{\xi_n\}_{n=0}^{\infty}$ for $x\sin(1/x)$ near 0.

Figure 6.1 plots the graphs of $y = x$ and $y = x\sin(1/x)$ near $x = 0$ to illustrate how the function $x \mapsto x$ approaches 0 monotonically while the function $x \mapsto x\sin(1/x)$ approaches 0 in an oscillatory fashion as $x \to 0$. From these graphs, it is readily apparent that x is locally nonvanishing near 0, whereas $x\sin(1/x)$ is sequentially nonvanishing near 0 but not locally nonvanishing near 0.

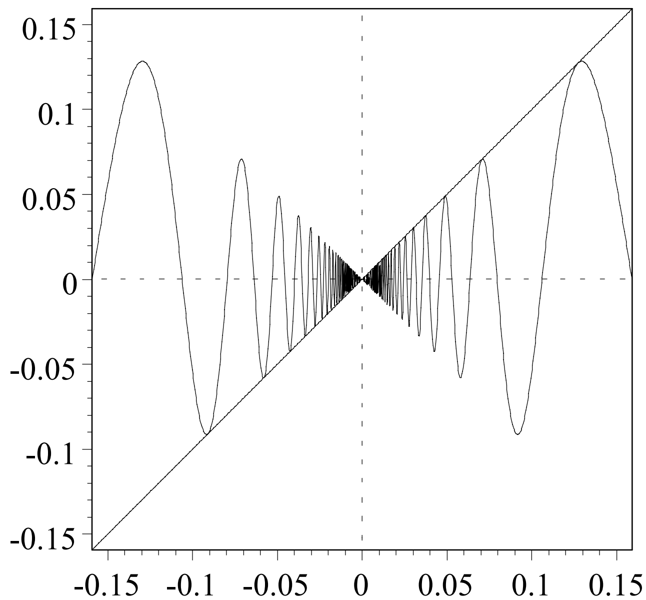

Figure 6.1: The Graphs of $y = x$ and $y = x\sin(1/x)$ near $x = 0$

What do we learn from these two examples? Although it is certainly easier to apply Proposition 6.2 over the special class $\mathrm{LNV}_{x_0}(X)$, we can tackle more difficult problems using Proposition 6.9 over the general class $\mathrm{SNV}_{x_0}(X)$. This trade-off between simplicity and generality is typical of the trade-offs that often confront us in mathematics.

In closing, we note that for arbitrary functions $f, g : X \to \mathbb{R}$, the relation $f(x) = o(g(x))$ as $x \to x_0$ implies the relation $f(x) = O(g(x))$ as $x \to x_0$; hence, the o relation is stronger than the O relation. The next section introduces a useful new asymptotic order relation which plays an intermediate role—it is weaker than the o relation but stronger than the O relation!

6.3 A New Asymptotic Order Relation

This section introduces the author's own $\tilde{o}_L$ order relation with constant L, which is weaker than the Landau o relation but stronger than the Landau O relation. The $\tilde{o}_L$ relation simplifies certain aspects of standard asymptotic analysis in one real variable.

Assume that $X \subset \overline{\mathbb{R}}$, and let x_0 be a limit point of X. Given arbitrary functions $f, g : X \to \mathbb{R}$ and an arbitrary constant $L \in \mathbb{R}$, and we say that $f(x)$ **is "little oh tilde" of** $g(x)$ **with constant** L **as** x **approaches** x_0 if

$$f(x) - L \cdot g(x) = o(g(x)) \quad \text{as} \quad x \to x_0.$$

We denote this relation by

$$f(x) = \tilde{o}_L(g(x)) \quad \text{as} \quad x \to x_0.$$

When $L = 0$, the relation $\tilde{o}_L$ reduces to the Landau o relation.

In the simplest interesting case, namely $g \in \mathrm{LNV}_{x_0}(X)$, the relation $f(x) = \tilde{o}_L(g(x))$ as $x \to x_0$ says that the quotient $f(x)/g(x)$ approaches the limit L as $x \to x_0$. This result follows immediately from Proposition 6.2, and is expressed formally as follows:

Proposition 6.10 *Let $X \subset \overline{\mathbb{R}}$ and let x_0 be a limit point of X. If $f : X \to \mathbb{R}$ is an arbitrary function and $g \in \mathrm{LNV}_{x_0}(X)$, then the relation*

$$f(x) = \tilde{o}_L(g(x)) \quad \textit{as} \quad x \to x_0$$

holds for the real constant L if and only if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = L.$$

It follows that the relation holds if and only if the limit exists, in which case the constant L is uniquely determined.

Under the hypothesis $g \in \mathrm{LNV}_{x_0}(X)$, the *uniqueness* of the constant L in the relation $f(x) = \tilde{o}_L(g(x))$ as $x \to x_0$ follows from the uniqueness of the value of the limit. The constant L is *not* unique under the hypothesis $g \in \mathrm{LV}_{x_0}(X)$, however. We address this case with the following proposition, which follows immediately from Proposition 6.4:

Proposition 6.11 *Let $X \subset \overline{\mathbb{R}}$ and let x_0 be a limit point of X. If $f : X \to \mathbb{R}$ is an arbitrary function and $g \in \mathrm{LV}_{x_0}(X)$, then the relation*

$$f(x) = \tilde{o}_L(g(x)) \quad \textit{as} \quad x \to x_0$$

holds for the real constant L if and only if $f \in \mathrm{LV}_{x_0}(X)$. When the relation holds under these hypotheses, it holds for every real number L.

Under the hypothesis $f, g \in \mathrm{LV}_{x_0}(X)$, the relation

$$f(x) - L \cdot g(x) = o(g(x)) \quad \text{as} \quad x \to x_0$$

in essence reduces to

$$0 - L \cdot 0 = o(0) \quad \text{as} \quad x \to x_0,$$

which clearly holds for all real numbers L. That is the reason for the nonuniqueness of L in this trivial case.

The hypothesis $g \in \mathrm{SNV}_{x_0}(X)$ is the most general condition that ensures the uniqueness of the constant L in the $\tilde{o}_L$ relation. This result follows immediately from Proposition 6.9, and is expressed formally as follows:

Proposition 6.12 *Let $X \subset \overline{\mathbb{R}}$ and let x_0 be a limit point of X. If $f : X \to \mathbb{R}$ is an arbitrary function and $g \in \mathrm{SNV}_{x_0}(X)$, then the relation*

$$f(x) = \tilde{o}_L(g(x)) \quad \textit{as} \quad x \to x_0$$

holds for the real constant L if and only if

$$\text{a-supp}\, f \subset_{x_0} \text{a-supp}\, g \quad \textit{and} \quad \lim_{\substack{x \rightsquigarrow x_0 \\ g}} \frac{f(x)}{g(x)} = L.$$

It follows that the relation holds if and only if the local containment holds and the supporting sequential limit exists, in which case the constant L is uniquely determined.

Propositions 6.11 and 6.12 together yield the following *complete characterization of the uniqueness of the constant L in the relation $\tilde{o}_L$:*

Corollary 6.13 *Let $X \subset \overline{\mathbb{R}}$ and let x_0 be a limit point of X. Assume that $f, g : X \to \mathbb{R}$ are arbitrary functions satisfying $f(x) = \tilde{o}_L(g(x))$ as $x \to x_0$ for at least one real number L. The value of L is unique if and only if $g \in \mathrm{SNV}_{x_0}(X)$.*

This corollary establishes that the condition $g \in \mathrm{SNV}_{x_0}(X)$ *is the weakest possible hypothesis that ensures the uniqueness of the constant L in the relation $\tilde{o}_L$*. Because of this, the class $\mathrm{SNV}_{x_0}(X)$ will play a fundamental role in our subsequent development of asymptotic theory.

Note that if $f, g \in \mathbb{R}^X$ are arbitrary functions, the relation $f(x) = o(g(x))$ as $x \to x_0$ implies the relation $f(x) = \tilde{o}_L(g(x))$ as $x \to x_0$ with constant $L = 0$. In turn, the relation $f(x) = \tilde{o}_L(g(x))$ as $x \to x_0$ with any value of L implies the relation $f(x) = O(g(x))$ as $x \to x_0$. We conclude that the $\tilde{o}_L$ order relation is weaker than the o relation but stronger than the O relation. The relative strengths of the o, $\tilde{o}_L$, and O order relations can be explained, in essence, by the following observation: Convergence to zero implies convergence to something finite, which in turn implies boundedness.

Finally, the notation we have chosen suggests that the $\tilde{o}_L$ relation is an *extension* of the o relation in some sense. To understand how this is so, recall that in set theory, a relation R on the set $\mathbb{R}^X$ is formally defined as a subset $R \subset \mathbb{R}^X \times \mathbb{R}^X$, and an *extension* of the relation R is simply another relation $\tilde{R} \subset \mathbb{R}^X \times \mathbb{R}^X$ such that $R \subset \tilde{R}$. If we use the o and $\tilde{o}_L$ relations to define two formal relations on $\mathbb{R}^X$ by

$$\begin{aligned} R &:= \{(f,g) \in \mathbb{R}^X \times \mathbb{R}^X : f(x) = o(g(x)) \text{ as } x \to x_0\} \\ \tilde{R} &:= \{(f,g) \in \mathbb{R}^X \times \mathbb{R}^X : f(x) = \tilde{o}_L(g(x)) \text{ as } x \to x_0 \text{ for some } L \in \mathbb{R}\}, \end{aligned}$$

the inclusion $R \subset \tilde{R}$ shows that the $\tilde{o}_L$ relation is a true extension of the o relation in the sense of set theory. In the same way, the O relation is a true extension of the $\tilde{o}_L$ relation in the sense of set theory. The usefulness of the intermediate $\tilde{o}_L$ relation will become clear as we continue our study of asymptotics.

6.4 Asymptotic Expansions in One Real Variable

Asymptotic sequences, asymptotic series, and asymptotic expansions in the sense of Poincaré are traditionally developed using both of the Landau O and o order relations defined earlier. The author prefers to develop this material using the o relation almost exclusively for the sake of simplicity; we can simplify matters even further by reformulating certain o conditions in terms of the author's $\tilde{o}_L$ relation. We will, however, use the O relation later on to describe the behavior of the remainders of these series expansions since O conveys more specific information than o in the context of remainder theory.

Let $X \subset \overline{\mathbb{R}}$, let x_0 be a limit point of X, and let $\{g_i\}_{i=0}^{n-1} \subset \mathbb{R}^X$, where *$n$ is either a* *positive integer or* ∞. We say that $\{g_i(x)\}_{i=0}^{n-1}$ is an **asymptotic sequence as** $x \to x_0$ if

$$g_{i+1}(x) = o(g_i(x)) \quad \text{as} \quad x \to x_0 \quad \text{for} \quad 0 \le i < n-1.$$

Proposition 6.2 allows us to reformulate this condition in terms of ordinary limits whenever $\{g_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{x_0}(X)$. Similarly, Proposition 6.9 allows us to reformulate this condition in terms of supporting sequential limits whenever $\{g_i\}_{i=0}^{n-1} \subset \mathrm{SNV}_{x_0}(X)$. When $n = 1$, the above condition is vacuous, which means that every one-term sequence $\{g_0(x)\}$ is an asymptotic sequence as $x \to x_0$.

Example 6.14 *The sequences*

$$\{x^n\}_{n=0}^{\infty} \subset \mathrm{LNV}_0(\mathbb{R}) \quad \textit{and} \quad \left\{\left(x \sin\frac{1}{x}\right)^n\right\}_{n=0}^{\infty} \subset \mathrm{SNV}_0(\mathbb{R}\backslash\{0\})$$

are both asymptotic sequences as $x \to 0$. *The sequence*

$$\left\{\frac{1}{x^n}\right\}_{n=0}^{\infty} \subset \mathrm{LNV}_{\infty}(\mathbb{R}\backslash\{0\})$$

is an asymptotic sequence as $x \to \infty$.

We will now prove the following important theorem about asymptotic sequences, *and will do so at the maximal level of generality possible in the function space* $\mathbb{R}^X$:

Theorem 6.15 *Let* $X \subset \overline{\mathbb{R}}$ *and let* x_0 *be a limit point of* X. *Assume that* $\{g_i\}_{i=0}^{n-1} \subset \mathrm{SNV}_{x_0}(X)$, *where* n *is either a positive integer or* ∞. *If* $\{g_i(x)\}_{i=0}^{n-1}$ *is an asymptotic sequence as* $x \to x_0$, *then* $\{g_i\}_{i=0}^{n-1}$ *is a linearly independent set in the function space* $\mathbb{R}^X$.

Proof. Since $\{g_i\}_{i=0}^{\infty}$ is a linearly independent set if and only if $\{g_i\}_{i=0}^{n-1}$ is a linearly independent set for all natural numbers $n \ge 1$, it suffices to show that $\{g_i\}_{i=0}^{n-1}$ is a linearly independent set in the case where n is finite. Let $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}$ be arbitrary. For $1 \le i \le n$, let P_i denote the proposition that $c_j = 0$ for $0 \le j < i$. We must show that the identity

$$\sum_{j=0}^{n-1} c_j \cdot g_j(x) = 0 \quad \text{for all} \quad x \in X \tag{6.1}$$

implies proposition P_n. We will prove this by finite induction on i.

Since $\{g_i(x)\}_{i=0}^{n-1}$ is an asymptotic sequence as $x \to x_0$, the transitivity of the o relation implies that

$$g_j(x) = o(g_i(x)) \quad \text{as} \quad x \to x_0 \quad \text{for} \quad 0 \le i < j < n.$$

Proposition 6.9 further implies that

$$\lim_{\substack{x \rightsquigarrow x_0 \\ g_i}} \frac{g_j(x)}{g_i(x)} = 0 \quad \text{for} \quad 0 \le i < j < n. \tag{6.2}$$

Dividing equation (6.1) by $g_0(x)$ yields

$$c_0 + \sum_{j=1}^{n-1} c_j \cdot \frac{g_j(x)}{g_0(x)} = 0 \quad \text{for all} \quad x \in \text{a-supp}\, g_0. \tag{6.3}$$

Taking the supporting sequential limit of equation (6.3) as $x \underset{g_0}{\rightsquigarrow} x_0$ and applying equation (6.2) yields $c_0 = 0$. This shows that proposition P_1 holds.

Now suppose that proposition P_i holds for arbitrary i with $1 \le i < n$. Under this hypothesis, dividing equation (6.1) by $g_i(x)$ yields

$$c_i + \sum_{j=i+1}^{n-1} c_j \cdot \frac{g_j(x)}{g_i(x)} = 0 \quad \text{for all} \quad x \in \text{a-supp}\, g_i. \tag{6.4}$$

Taking the supporting sequential limit of equation (6.4) as $x \underset{g_i}{\rightsquigarrow} x_0$ and applying equation (6.2) yields $c_i = 0$. This shows that proposition P_{i+1} holds.

We have shown that proposition P_1 holds, and that proposition P_i implies that proposition P_{i+1} holds whenever $1 \le i < n$. It follows by finite induction on i that proposition P_i holds for $1 \le i \le n$. In particular, proposition P_n holds, as desired. ■

If $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}$ consists of arbitrary constants and $\{g_i(x)\}_{i=0}^{n-1}$ is an asymptotic sequence as $x \to x_0$, the sum

$$\sum_{i=0}^{n-1} c_i \cdot g_i(x) \tag{6.5}$$

is called an **asymptotic series as** $x \to x_0$. If $n = \infty$, we treat the resulting infinite sum as a formal series since asymptotic series are permitted to diverge. We denote the partial sums of this series by

$$s_m := \sum_{i=0}^{m-1} c_i \cdot g_i \quad \text{for} \quad 0 \le m < n+1.$$

Since the empty sum is zero by convention, we define $s_0 := 0$.

Given an arbitrary function $f : X \to \mathbb{R}$, we define the remainders of f with respect to asymptotic series (6.5) by

$$r_m := f - s_m \quad \text{for} \quad 0 \le m < n+1.$$

Note in particular that $r_0 := f$.

Remark 6.16 *We will make heavy use of this partial sum and remainder notation since it simplifies matters considerably!*

We say that the asymptotic series (6.5) is an **asymptotic expansion of** $f(x)$ **to** n **terms as** $x \to x_0$ if

$$r_{i+1}(x) = o(g_i(x)) \quad \text{as} \quad x \to x_0 \quad \text{for} \quad 0 \le i < n, \tag{6.6}$$

and we write

$$f(x) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x) \quad \text{as} \quad x \to x_0.$$

If we substitute

$$r_{i+1} := f - s_{i+1} = f - (s_i + c_i \cdot g_i) = r_i - c_i \cdot g_i$$

into condition (6.6), we obtain the equivalent condition

$$r_i(x) - c_i \cdot g_i(x) = o(g_i(x)) \quad \text{as} \quad x \to x_0 \quad \text{for} \quad 0 \le i < n,$$

which we can reformulate in terms of the $\tilde{o}_L$ relation with constant $L = c_i$ as

$$r_i(x) = \tilde{o}_{c_i}(g_i(x)) \quad \text{as} \quad x \to x_0 \quad \text{for} \quad 0 \le i < n. \tag{6.7}$$

This reformulation and Proposition 6.10 together yield the following standard result:

Proposition 6.17 *Let $X \subset \overline{\mathbb{R}}$ and let x_0 be a limit point of X. Let $f : X \to \mathbb{R}$ be an arbitrary function and assume that $\{g_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{x_0}(X)$, where n is either a positive integer or ∞. If $\{g_i(x)\}_{i=0}^{n-1}$ is an asymptotic sequence as $x \to x_0$, then the asymptotic expansion*

$$f(x) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x) \quad \textit{as} \quad x \to x_0$$

holds with the coefficients $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}$ *if and only if*

$$\lim_{x \to x_0} \frac{r_i(x)}{g_i(x)} = c_i \quad \textit{for} \quad 0 \leq i < n.$$

It follows that $f(x)$ *has an asymptotic expansion to* n *terms as* $x \to x_0$ *with respect to the asymptotic sequence* $\{g_i(x)\}_{i=0}^{n-1}$ *if and only if all of these limits exists, in which case the coefficients* $\{c_i\}_{i=0}^{n-1}$ *are uniquely determined.*

Applying the more powerful Proposition 6.12 to the reformulation (6.7) of condition (6.6) yields the following more general result:

Proposition 6.18 *Let* $X \subset \overline{\mathbb{R}}$ *and let* x_0 *be a limit point of* X*. Let* $f : X \to \mathbb{R}$ *be an arbitrary function and assume that* $\{g_i\}_{i=0}^{n-1} \subset \mathrm{SNV}_{x_0}(X)$*, where* n *is either a positive integer or* ∞*. If* $\{g_i(x)\}_{i=0}^{n-1}$ *is an asymptotic sequence as* $x \to x_0$*, then the asymptotic expansion*

$$f(x) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x) \quad \textit{as} \quad x \to x_0$$

holds with the coefficients $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}$ *if and only if*

$$\text{a-supp}\, r_i \subset_{x_0} \text{a-supp}\, g_i \quad \textit{and} \quad \lim_{\substack{x \rightsquigarrow x_0 \\ g_i}} \frac{r_i(x)}{g_i(x)} = c_i \quad \textit{for} \quad 0 \leq i < n.$$

It follows that $f(x)$ *has an asymptotic expansion to* n *terms as* $x \to x_0$ *with respect to the asymptotic sequence* $\{g_i(x)\}_{i=0}^{n-1}$ *if and only if all of these local containments hold and all of these supporting sequential limits exist, in which case the coefficients* $\{c_i\}_{i=0}^{n-1}$ *are uniquely determined.*

We conclude this section by using asymptotic expansions to define a familiar relation on the set $\mathbb{R}^X$: Given arbitrary $f, g \in \mathbb{R}^X$, we say that $f(x)$ **is asymptotically equivalent to** $g(x)$ **as** $x \to x_0$ if the following one-term asymptotic expansion holds:

$$f(x) \sim g(x) \quad \text{as} \quad x \to x_0.$$

By definition, this asymptotic expansion holds if and only if

$$f(x) - g(x) = o(g(x)) \quad \text{as} \quad x \to x_0,$$

which we can reformulate in terms of the $\tilde{o}_L$ relation with constant $L = 1$ as

$$f(x) = \tilde{o}_1(g(x)) \quad \text{as} \quad x \to x_0.$$

By applying the definition of the o relation with $\epsilon = \frac{1}{2}$, we can deduce from the o relation above that f and g have the same zeros inside some deleted neighborhood $(X \cap N_\epsilon)\backslash\{\, x_0\}$; therefore, f and g also have the same algebraic supports inside this deleted neighborhood, which means that

$$\text{a-supp}\, f =_{x_0} \text{a-supp}\, g.$$

This innocuous-looking local equality has the following intriguing consequences:

$$\begin{aligned}
&f \in \mathrm{LV}_{x_0}(X) && \text{if and only if} && g \in \mathrm{LV}_{x_0}(X),\\
&f \in \mathrm{SNV}_{x_0}(X) && \text{if and only if} && g \in \mathrm{SNV}_{x_0}(X),\\
&f \in \mathrm{LNV}_{x_0}(X) && \text{if and only if} && g \in \mathrm{LNV}_{x_0}(X).
\end{aligned}$$

These consequences imply that asymptotic equivalence on $\mathbb{R}^X$ induces three distinct relations on the subsets $\mathrm{LV}_{x_0}(X)$, $\mathrm{SNV}_{x_0}(X)$, and $\mathrm{LNV}_{x_0}(X)$ by restriction. We will now consider each relation in turn:

1. If $f, g \in \mathrm{LV}_{x_0}(X)$, then the relation $f(x) \sim g(x)$ as $x \to x_0$ always holds by Proposition 6.11. This shows that asymptotic equivalence is trivially an equivalence relation on $\mathrm{LV}_{x_0}(X)$.

2. If $f, g \in \mathrm{SNV}_{x_0}(X)$, then the relation $f(x) \sim g(x)$ as $x \to x_0$ holds if and only if

   $$\text{a-supp}\, f =_{x_0} \text{a-supp}\, g \quad \text{and} \quad \lim_{\substack{x \rightsquigarrow x_0 \\ g}} \frac{f(x)}{g(x)} = 1,$$

   by the above analysis and Proposition 6.12. We say that $\{\xi_n\}_{n=0}^{\infty}$ is **eventually a supporting sequence for g near x_0** if $\{\xi_n\}_{n=N}^{\infty}$ is a supporting sequence for g near x_0 for some natural number N. If we replace supporting sequences by eventually supporting sequences in the definition of the supporting sequential limit, we do not alter either the conditions under which the limit exists or the value of the limit when it does exist. Furthermore, the local equality a-supp $f =_{x_0}$ a-supp g implies that f and g have exactly the same eventually supporting sequences near x_0. This means that a supporting sequential limit as $x \underset{f}{\rightsquigarrow} x_0$ is also a supporting sequential limit as $x \underset{g}{\rightsquigarrow} x_0$, and conversely. These facts, together with the above reformulation, are sufficient to establish that asymptotic equivalence is an equivalence relation on $\mathrm{SNV}_{x_0}(X)$.

3. If $f, g \in \mathrm{LNV}_{x_0}(X)$, then the relation $f(x) \sim g(x)$ as $x \to x_0$ holds if and only if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1,$$

by Proposition 6.10. Asymptotic equivalence is an equivalence relation on $\mathrm{LNV}_{x_0}(X)$ by restriction of the previous equivalence relation on the superset $\mathrm{SNV}_{x_0}(X)$; we can also establish this result directly from the above reformulation using the properties of ordinary limits. This case is the most familiar of the three and is what is traditionally meant by asymptotic equivalence—we have chosen to broaden the definition here in order to illustrate the power of the general theory that we have developed.

Remark 6.19 *Since asymptotic equivalence is an equivalence relation on both* $\mathrm{LV}_{x_0}(X)$ *and* $\mathrm{SNV}_{x_0}(X)$*, it is also an equivalence relation on their union, which is* $\mathbb{R}^X$.

This concludes our nontraditional introduction to traditional asymptotic methods in one real variable. We have adopted this novel approach in order to prepare the way for a radical reformulation of these standard methods in a more abstract setting based on the author's own "generalized inner product spaces." This original abstraction unifies many seemingly unrelated notions from asymptotic analysis, interpolation theory, and of course classical linear algebra. We will begin to develop this unifying abstraction in Chapter 11; meanwhile, the next chapter begins our study of the author's original asymptotic methods in two real variables.

Chapter 7

The Theory of Dual Asymptotic Expansions

This chapter develops a new approach to asymptotic analysis for real-valued functions of two real variables. These original asymptotic methods are based on the author's own dual asymptotic expansions and asymptotic splitting operator; in fact, this chapter presents a vastly new-and-improved approach which both simplifies and extends the original asymptotic methods first presented in the author's master's thesis [Cha98].

In the first section of this chapter, we will define a useful new kind of limit for functions of two real variables, and two useful new composition operators. In the second section, we will use these three original constructions to define and study the asymptotic splitting operator in full generality; this study will culminate in a complete characterization of the structure of the operator. In the third and final section of this chapter, we will use the structure theory of the asymptotic splitting operator to develop a definitive uniqueness and existence theory for dual asymptotic expansions.

This chapter includes many concrete examples which generally serve two purposes: These examples illustrate how to apply the new constructions of this chapter. These examples also shed light on the relationships between various hypotheses, often proving by example that two hypotheses are independent—that neither hypothesis follows as a consequence of the other. This assures us that these hypotheses are not redundant, but are indeed truly necessary for our work in this area of asymptotic analysis.

7.1 New Operations on Functions of Two Real Variables

This section contains a careful study of the kind of limiting operations which arise in connection with the asymptotic splitting operator. In the first subsection, we will explore a new kind of bivariate limit which is constructed from two iterated univariate limits which are independent of order. In the second subsection, we will develop ways to use univariate limits to compose two given bivariate functions in order to define a new bivariate function.

7.1.1 Rectilinear Limits versus Classical Limits

This subsection develops a new kind of limit for real-valued functions of two real variables. Let us begin with the following formal definition:

Definition 7.1 *Let $X, Y \subset \overline{\mathbb{R}}$ be arbitrary subsets, and let x_0 and y_0 be limit points of X and Y, respectively. Let $f : X \times Y \to \mathbb{R}$ be an arbitrary function. Consider the following iterated one-dimensional limits:*

$$L_1 := \lim_{x \to x_0} \left(\lim_{y \to y_0} f(x, y) \right) \quad \textit{and} \quad L_2 := \lim_{y \to y_0} \left(\lim_{x \to x_0} f(x, y) \right).$$

If both iterated limits exist and are equal, we call their common value the ***rectilinear limit of*** $f(x, y)$ ***as*** $\{x \to x_0,\ y \to y_0\}$*, and we write*

$$\lim_{\{x \to x_0,\ y \to y_0\}} f(x, y) = L_1 = L_2.$$

The essential idea embodied by the rectilinear limit is that *the value of the iterated one-dimensional limits is independent of the order in which the one-dimensional limits are evaluated.* This independence of order is the motivation for the use of set notation, which is also independent of order:

$$\{x \to x_0,\ y \to y_0\} \quad = \quad \{y \to y_0,\ x \to x_0\}.$$

How does the rectilinear limit as $\{x \to x_0,\ y \to y_0\}$ get its name? The word "rectilinear" refers to motion occurring along lines that are parallel to the coordinate axes. For example, if we hold x fixed and let $y \to y_0$, the point (x, y) in the inner one-dimensional limit

$$\lim_{y \to y_0} f(x, y)$$

approaches the point (x, y_0) along a line parallel to the y-axis. Similarly, if we hold y fixed and let $x \to x_0$, the point (x, y) in the inner one-dimensional limit

$$\lim_{x \to x_0} f(x, y)$$

approaches the point (x_0, y) along a line parallel to the x-axis.

For given values of x and y, we can use these two inner one-dimensional limits to define two new univariate functions by the formulas

$$g(x) := \lim_{y \to y_0} f(x, y) \quad \text{and} \quad h(y) := \lim_{x \to x_0} f(x, y).$$

If these two one-dimensional limits exist for all x near x_0 and all y near y_0, then the functions g and h are well-defined in a deleted neighborhood of x_0 and a deleted neighborhood of y_0, respectively. If in turn the outer one-dimensional limits exist and satisfy

$$\lim_{x \to x_0} g(x) = \lim_{y \to y_0} h(y),$$

then by definition, the rectilinear limit of $f(x, y)$ as $\{x \to x_0,\ y \to y_0\}$ also exists and satisfies

$$\lim_{\{x \to x_0,\ y \to y_0\}} f(x, y) = \lim_{x \to x_0} g(x) = \lim_{y \to y_0} h(y).$$

If, however, the first inner one-dimensional limit does *not* yield a well-defined function g in *any* deleted neighborhood of x_0, or the second inner one-dimensional limit does *not* yield a well-defined function h in *any* deleted neighborhood of y_0, then the outer one-dimensional limits are ill-posed, and the rectilinear limit of $f(x, y)$ as $\{x \to x_0,\ y \to y_0\}$ therefore fails to exist.

The following example demonstrates that the rectilinear limit as $\{x \to x_0,\ y \to y_0\}$ can exist even when the classical limit as $(x, y) \to (x_0, y_0)$ fails to exist:

Example 7.2 *Let* $X := Y := \mathbb{R}\backslash\{0\}$ *and* $x_0 := y_0 := 0$, *and note that* 0 *is a limit point of* $\mathbb{R}\backslash\{0\}$. *Consider the function* $f : X \times Y \to \mathbb{R}$ *defined by*

$$f(x, y) := \frac{xy}{x^2 + y^2} \quad \textit{for all} \quad (x, y) \in X \times Y.$$

First, calculate the following iterated one-dimensional limits:

$$L_1 := \lim_{x\to 0}\left(\lim_{y\to 0}\frac{xy}{x^2+y^2}\right) = \lim_{x\to 0}\frac{0}{x^2} = \lim_{x\to 0} 0 = 0$$

$$L_2 := \lim_{y\to 0}\left(\lim_{x\to 0}\frac{xy}{x^2+y^2}\right) = \lim_{y\to 0}\frac{0}{y^2} = \lim_{y\to 0} 0 = 0.$$

Since $L_1 = 0 = L_2$*, the rectilinear limit of* $f(x,y)$ *as* $\{x\to 0,\ y\to 0\}$ *exists, and we write*

$$\lim_{\{x\to 0,\ y\to 0\}}\frac{xy}{x^2+y^2} = 0.$$

Now calculate the limit of $f(x,y)$ *as* $(x,y)\to(0,0)$ *along the two different lines* $y = x$ *and* $y = -x$*:*

$$L_3 := \lim_{x\to 0} f(x,x) = \lim_{x\to 0}\frac{x^2}{2x^2} = \lim_{x\to 0}\frac{1}{2} = \frac{1}{2}$$

$$L_4 := \lim_{x\to 0} f(x,-x) = \lim_{x\to 0}\frac{-x^2}{2x^2} = \lim_{x\to 0}\frac{-1}{2} = -\frac{1}{2}.$$

Since $L_3 = \frac{1}{2} \neq -\frac{1}{2} = L_4$*, the classical limit of* $f(x,y)$ *as* $(x,y)\to(0,0)$ *fails to exist.*

Conversely, the following example demonstrates that the classical limit as $(x,y)\to(x_0,y_0)$ can exist even when the rectilinear limit as $\{x\to x_0,\ y\to y_0\}$ fails to exist:

Example 7.3 *Let* $X := Y := \mathbb{R}$ *and* $x_0 := y_0 := 0$*, and note that* 0 *is a limit point of* $\mathbb{R}$*. Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$f(x,y) := \left(x^2+y^2\right)\cdot\chi_{(0,\infty)}(x)\cdot\chi_{(0,\infty)}(y) \quad \textit{for all} \quad (x,y)\in\mathbb{R}^2,$$

where $\chi_{(0,\infty)}$ *denotes the characteristic function of the open interval* $(0,\infty)$*. Since*

$$0 \le f(x,y) \le x^2+y^2 \quad \textit{for all} \quad (x,y)\in\mathbb{R}^2,$$

the classical limit of $f(x,y)$ *as* $(x,y)\to(0,0)$ *exists and satisfies*

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

In contrast, for every real $x > 0$, the following one-sided, one-dimensional limits yield

$$\lim_{y \to 0^+} f(x, y) = \lim_{y \to 0^+} \left(x^2 + y^2\right) = x^2 \neq 0 = \lim_{y \to 0^-} 0 = \lim_{y \to 0^-} f(x, y).$$

This shows that the two-sided, one-dimensional limit

$$\lim_{y \to 0} f(x, y)$$

does not exist whenever $x > 0$; consequently, the limit does not produce a well-defined function of x in any deleted neighborhood of 0. Since f is a symmetric function, a similar argument shows that the one-dimensional limit

$$\lim_{x \to 0} f(x, y)$$

does not exist whenever $y > 0$ and therefore fails to produce a well-defined function of y in any deleted neighborhood of 0. Consequently, both of the iterated one-dimensional limits

$$\lim_{x \to 0} \left(\lim_{y \to 0} f(x, y) \right) \quad \text{and} \quad \lim_{y \to 0} \left(\lim_{x \to 0} f(x, y) \right)$$

are ill-posed, which means that the rectilinear limit of $f(x, y)$ as $\{x \to 0,\ y \to 0\}$ fails to exist.

The previous two examples demonstrate unequivocally that the rectilinear limit and the classical limit are distinct constructions. Nevertheless, there is a simple condition which ensures that the rectilinear limit and the classical limit exist simultaneously and yield the same value. The following proposition states this condition precisely:

Proposition 7.4 *Let $X, Y \subset \overline{\mathbb{R}}$ be intervals, let $(x_0, y_0) \in X \times Y$, and let $f : X \times Y \to \mathbb{R}$ be an arbitrary function. If f is continuous in a neighborhood of (x_0, y_0), then the rectilinear limit of $f(x, y)$ as $\{x \to x_0,\ y \to y_0\}$ and the classical limit of $f(x, y)$ as $(x, y) \to (x_0, y_0)$ both exist and satisfy*

$$\lim_{\{x \to x_0,\ y \to y_0\}} f(x, y) = f(x_0, y_0) = \lim_{(x,y) \to (x_0, y_0)} f(x, y).$$

Note that it is not enough for f to be continuous merely *at* the point (x_0, y_0), for this would ensure only that the classical limit exists, not that the rectilinear limit exists. In order to ensure that the rectilinear limit exists, we need a stronger hypothesis, such as the continuity of f in a *neighborhood* of (x_0, y_0).

This completes our study of the rectilinear limit, which is based on two iterated univariate limits. In the next subsection, we will use one univariate limit at a time to develop two new kinds of limiting operations for bivariate functions.

7.1.2 Covariant and Contravariant Asymptotic Composition Operators

The limiting operations that we will develop in this subsection will ultimately prove their value to us by unlocking all the secrets of existence for both the asymptotic splitting operator and dual asymptotic expansions! Let us begin with the following terminology and notation, which provides another useful hypothesis for the asymptotic analysis of functions of two real variables:

Definition 7.5 *Let $X, Y \subset \overline{\mathbb{R}}$ be arbitrary subsets, and let x_0 and y_0 be limit points of X and Y, respectively. Let $f : X \times Y \to \mathbb{R}$ be an arbitrary function. If the cross-sections of the bivariate function f satisfy*

$$f^y \in \mathrm{LNV}_{x_0}(X) \quad \text{for all} \quad y \in Y \quad \text{and} \quad f_x \in \mathrm{LNV}_{y_0}(Y) \quad \text{for all} \quad x \in X,$$

*we say that f **is locally nonvanishing near the lines** $x = x_0$ **and** $y = y_0$. We denote the set of all such real-valued functions on $X \times Y$ by $\mathrm{LNV}_{x_0}^{y_0}(X \times Y)$.*

The condition $f \in \mathrm{LNV}_{x_0}^{y_0}(X \times Y)$ ensures that the reciprocal $1/f(x, y)$ is defined for each $y \in Y$ and all x sufficiently close to x_0, as well as for each $x \in X$ and all y sufficiently close to y_0. To put it another way, this hypothesis ensures that $1/f(x, y)$ is defined as x varies locally while y varies globally, and as y varies locally while x varies globally. This combination of local and global behavior forms the weakest hypothesis that will allow us to construct globally-defined asymptotic compositions of bivariate functions. We now define these asymptotic composition operators formally:

Definition 7.6 *Let $X, Y, X', Y' \subset \overline{\mathbb{R}}$ be arbitrary subsets, and let x_0 and y_0 be limit points of X and Y, respectively. Assume that $f_1 \in \mathrm{LNV}_{x_0}^{y_0}(X \times Y)$, and define the asymptotic composition operators $\circ_{x_0}$ and $\circ^{y_0}$ as follows:*

1. *If $f_2 : X \times Y' \to \mathbb{R}$ is an arbitrary function, we can use a one-dimensional limit as $x \to x_0$ to define a new function $f_1 \circ_{x_0} f_2 : Y \times Y' \to \mathbb{R}$ by the formula*

$$(f_1 \circ_{x_0} f_2)(y_1, y_2) := \lim_{x \to x_0} \frac{f_2(x, y_2)}{f_1(x, y_1)} \quad \textit{for all} \quad (y_1, y_2) \in Y \times Y'. \tag{7.1}$$

   *We say that $f_1 \circ_{x_0} f_2$ **is well-defined** if the limit exists for all $(y_1, y_2) \in Y \times Y'$. We call $f_1 \circ_{x_0} f_2$ the **covariant asymptotic composition of** f_1 **and** f_2 **at** x_0, and we call $\circ_{x_0}$ the **covariant asymptotic composition operator at** x_0.*

2. *If $f_2 : X' \times Y \to \mathbb{R}$ is an arbitrary function, we can use a one-dimensional limit as $y \to y_0$ to define a new function $f_1 \circ^{y_0} f_2 : X' \times X \to \mathbb{R}$ by the formula*

$$(f_1 \circ^{y_0} f_2)(x_1, x_2) := \lim_{y \to y_0} \frac{f_2(x_1, y)}{f_1(x_2, y)} \quad \textit{for all} \quad (x_1, x_2) \in X' \times X. \tag{7.2}$$

   *We say that $f_1 \circ^{y_0} f_2$ **is well-defined** if the limit exists for all $(x_1, x_2) \in X' \times X$. We call $f_1 \circ^{y_0} f_2$ the **contravariant asymptotic composition of** f_1 **and** f_2 **at** y_0, and we call $\circ^{y_0}$ the **contravariant asymptotic composition operator at** y_0.*

The word "covariant" refers to the way that the indices $i, j \in \{1, 2\}$ in the expression $f_i(x, y_j)$ *vary together* in equation (7.1), while the word "contravariant" refers to the way that the indices in the expression $f_i(x_j, y)$ *vary in opposition* in equation (7.2). We have followed the established mathematical convention of denoting *covariant* objects using *subscripts* (e.g., covariant tensors, covariant functors) and *contravariant* objects using *superscripts* (e.g., contravariant tensors, contravariant functors). The following example illustrates a typical use of the covariant and contravariant asymptotic composition operators:

Example 7.7 *Let $X := Y := X' := Y' := (0, \frac{\pi}{2})$, and note that $x_0 := 0$ and $y_0 := \frac{\pi}{2}$ are limit points of the interval $(0, \frac{\pi}{2})$. Consider the function $f : (0, \frac{\pi}{2})^2 \to \mathbb{R}$ defined by*

$$f(x, y) := \sin(x + y) \quad \textit{for all} \quad (x, y) \in \left(0, \frac{\pi}{2}\right)^2.$$

Since $\sin(x+y) > 0$ for $0 < x+y < \pi$, it follows that $f \in \mathrm{LNV}_0^{\pi/2}\left((0, \frac{\pi}{2})^2\right)$. The covariant asymptotic composition of f with itself at 0 is well-defined, and is given by

$$(f \circ_0 f)(y_1, y_2) := \lim_{x \to 0^+} \frac{\sin(x + y_2)}{\sin(x + y_1)} = \frac{\sin y_2}{\sin y_1} \quad \textit{for all} \quad (y_1, y_2) \in \left(0, \frac{\pi}{2}\right)^2.$$

The contravariant asymptotic composition of f with itself at $\frac{\pi}{2}$ is well-defined, and is given by

$$(f \circ^{\pi/2} f)(x_1, x_2) := \lim_{y \to \pi/2-} \frac{\sin(x_1 + y)}{\sin(x_2 + y)} = \frac{\cos x_1}{\cos x_2} \quad \textit{for all} \quad (x_1, x_2) \in \left(0, \frac{\pi}{2}\right)^2.$$

Since the function value $f(x, y) := \sin(x+y)$ is certainly well-defined for all $(x, y) \in \mathbb{R}^2$, why did we restrict the domain of the function f to the square $(0, \frac{\pi}{2})^2$ in the previous example? We did this to ensure that the compositions $f \circ_0 f$ and $f \circ^{\pi/2} f$ are well-defined on $(0, \frac{\pi}{2})^2$. Restricting the domain in this way avoids division by zero on the boundary lines $y_1 = 0$ and $x_2 = \frac{\pi}{2}$, where $\sin y_1 = \cos x_2 = 0$.

We conclude this subsection by noting that the author has modeled the asymptotic composition operators after the following well-known composition operator for Hilbert-Schmidt kernels $f, g : [0, 1]^2 \to \mathbb{R}$ of Fredholm linear integral operators:

$$(f \circ g)(x, y) := \int_0^1 f(x, t)\, g(t, y)\, dt \quad \text{for all} \quad (x, y) \in [0, 1]^2.$$

We can rewrite this standard composition operator in terms of the classical inner product

$$\langle u, v \rangle := \int_0^1 u(t)\, v(t)\, dt$$

and the x- and y-sections of the kernels f and g as follows:

$$(f \circ g)(x, y) := \langle f_x, g^y \rangle \quad \text{for all} \quad (x, y) \in [0, 1]^2.$$

By analogy, if we define the "asymptotic inner products"

$$[u_1, u_2]_{x_0} := \lim_{x \to x_0} \frac{u_2(x)}{u_1(x)} \quad \text{and} \quad [v_1, v_2]^{y_0} := \lim_{y \to y_0} \frac{v_2(y)}{v_1(y)},$$

we can rewrite the defining equations (7.1) and (7.2) for the asymptotic composition operators in terms of asymptotic inner products and cross-sections in this way:

$$\begin{aligned}
(f_1 \circ_{x_0} f_2)(y_1, y_2) &= \lim_{x \to x_0} \frac{(f_2)^{y_2}(x)}{(f_1)^{y_1}(x)} = [(f_1)^{y_1}, (f_2)^{y_2}]_{x_0} \quad \text{for all} \quad (y_1, y_2) \in Y \times Y' \\
(f_1 \circ^{y_0} f_2)(x_1, x_2) &= \lim_{y \to y_0} \frac{(f_2)_{x_1}(y)}{(f_1)_{x_2}(y)} = [(f_1)_{x_2}, (f_2)_{x_1}]^{y_0} \quad \text{for all} \quad (x_1, x_2) \in X' \times X.
\end{aligned}$$

These similarities are much more than mere analogy! Although the asymptotic inner products $[\bullet,\bullet]_{x_0}$ and $[\bullet,\bullet]^{y_0}$ are clearly *not* inner products in the classical sense (due to nonlinearity in the first arguments), *both* the classical inner product *and* the asymptotic inner product are indeed inner products in exactly the same *generalized* sense. The author has unified classical inner product spaces and asymptotic inner product spaces by developing a simple collection of algebraic axioms for generalized inner product spaces; despite their simplicity, these axioms are flexible enough to accommodate partial nonlinearity in applications such as asymptotic analysis. We will introduce these axioms and consider their implications in Chapter 11.

7.2 The Asymptotic Splitting Operator in the General Case

In this section, we will define and study the asymptotic splitting operator in full generality using the rectilinear limit and the asymptotic composition operators developed in the previous section. Our general strategy will be to define the asymptotic splitting operator using the rectilinear limit, and then to use the asymptotic composition operators to characterize the structure of the operator so defined.

7.2.1 Definitions and Fundamental Results

The following definition provides one of two alternate hypotheses that will allow us to define the asymptotic splitting operator rigorously:

Definition 7.8 *Let $X, Y \subset \overline{\mathbb{R}}$ be arbitrary subsets, and let x_0 and y_0 be limit points of X and Y, respectively. Let $f : X \times Y \to \mathbb{R}$ be an arbitrary function. If there exist open neighborhoods M and N of x_0 and y_0, respectively, such that*

$$f(x,y) \neq 0 \quad \text{for all} \quad (x,y) \in [(X \cap M)\backslash\{x_0\}] \times [(Y \cap N)\backslash\{y_0\}]\,,$$

we say that f ***is locally nonvanishing near*** *(x_0, y_0). We denote the set of all such real-valued functions on $X \times Y$ by $\mathrm{LNV}_{(x_0,y_0)}(X \times Y)$.*

The hypothesis $f \in \mathrm{LNV}_{(x_0,y_0)}(X \times Y)$ asserts that f does not vanish in some neighborhood of (x_0, y_0), except possibly on the rectilinear coordinate lines $x = x_0$ and $y = y_0$. For our purposes, we do not care whether $f(x,y)$ is even *defined* on the lines $x = x_0$ and

$y = y_0$. As we shall see, it is the *limiting behavior* of f near these lines that is important to us.

The purpose of the hypothesis $f \in \mathrm{LNV}_{(x_0,y_0)}(X \times Y)$ is to ensure that the reciprocal $1/f(x,y)$ remains well-defined as x and y vary in a manner suitable for evaluating a rectilinear limit such as

$$\lim_{\{x \to x_0,\ y \to y_0\}} \frac{1}{f(x,y)}.$$

The *alternate* hypothesis $f \in \mathrm{LNV}_{x_0}^{y_0}(X \times Y)$ also ensures that the reciprocal $1/f(x,y)$ remains well-defined as x and y vary during the evaluation of this rectilinear limit. How do these two hypotheses differ? The first hypothesis imposes local behavior on f near the *point* (x_0, y_0), whereas the second hypothesis imposes local behavior on f near the *lines* $x = x_0$ *and* $y = y_0$. The first hypothesis is *a strictly local property*, whereas the second hypothesis is actually *a hybrid of local and global properties*. We will now use these various properties to formally define the asymptotic splitting operator $\Upsilon_{(x_0,y_0)}$:

Definition 7.9 *Let $X, Y \subset \overline{\mathbb{R}}$ be arbitrary subsets, and let x_0 and y_0 be limit points of X and Y, respectively. Assume that*

$$f \in \mathrm{LNV}_{(x_0,y_0)}(X \times Y) \cup \mathrm{LNV}_{x_0}^{y_0}(X \times Y).$$

Under this hypothesis, we can use a rectilinear limit as $\{x' \to x_0,\ y' \to y_0\}$ to define a new function

$$\Upsilon_{(x_0,y_0)} f : X \times Y \to \mathbb{R}$$

by the formula

$$\Upsilon_{(x_0,y_0)} f(x,y) := \lim_{\{x' \to x_0,\ y' \to y_0\}} \frac{f(x,y')\, f(x',y)}{f(x',y')} \quad \textit{for all} \quad (x,y) \in X \times Y.$$

*We say that the function $\Upsilon_{(x_0,y_0)} f$ **is well-defined** if the rectilinear limit exists for all $(x,y) \in X \times Y$. We call the operator $\Upsilon_{(x_0,y_0)}$ the **asymptotic splitting operator at** (x_0, y_0).*

The following proposition addresses the important special case of the asymptotic splitting operator that we studied earlier in Section 5.1:

Proposition 7.10 *Let $X, Y \subset \overline{\mathbb{R}}$ be intervals and let $(x_0, y_0) \in X \times Y$. Assume that $f \in C(X \times Y)$. If $f(x_0, y_0) \neq 0$, then the function $\Upsilon_{(x_0,y_0)} f$ is well-defined and satisfies*

$$\Upsilon_{(x_0,y_0)} f(x, y) = \frac{f(x, y_0)\, f(x_0, y)}{f(x_0, y_0)} \quad \textit{for all} \quad (x, y) \in X \times Y.$$

Proof. Since f is continuous at (x_0, y_0) and $f(x_0, y_0) \neq 0$, there exists an open rectangular neighborhood $M \times N$ of (x_0, y_0) such that

$$f(x, y) \neq 0 \quad \text{for all} \quad (x, y) \in (X \times Y) \cap (M \times N).$$

For convenience, let $X' := X \cap M$ and $Y' := Y \cap N$, and note that

$$(X \times Y) \cap (M \times N) = X' \times Y'.$$

In this new notation, we can write

$$f(x', y') \neq 0 \quad \text{for all} \quad (x', y') \in X' \times Y'.$$

This property allows us to define a new function

$$F : X' \times Y' \times X \times Y \to \mathbb{R}$$

with two variables x' and y' and two parameters x and y by the formula

$$F(x', y'; x, y) := \frac{f(x, y')\, f(x', y)}{f(x', y')} \quad \text{for all} \quad (x', y'; x, y) \in X' \times Y' \times X \times Y.$$

Since $f \in C(X \times Y)$, it follows that $F \in C(X' \times Y' \times X \times Y)$. Since $X', Y' \subset \overline{\mathbb{R}}$ are intervals and $(x_0, y_0) \in X' \times Y'$, Proposition 7.4 implies that

$$\Upsilon_{(x_0,y_0)} f(x, y) := \lim_{\{x' \to x_0,\ y' \to y_0\}} F(x', y'; x, y) = F(x_0, y_0; x, y) := \frac{f(x, y_0)\, f(x_0, y)}{f(x_0, y_0)}$$

for every choice of parameters $(x, y) \in X \times Y$. ■

Note that it is not enough to assume that f is continuous merely in a *neighborhood* of the point (x_0, y_0), for that would ensure only that the function $\Upsilon_{(x_0,y_0)} f$ is defined in a *neighborhood* of (x_0, y_0) rather than on all of $X \times Y$. In order to ensure that $\Upsilon_{(x_0,y_0)} f$ is *globally* defined, we need to ensure that $F(x', y'; x, y)$ is continuous in the variables (x', y')

as the parameters (x, y) vary *globally* over $X \times Y$. This, in turn, requires a *global* hypothesis, such as $f \in C(X \times Y)$.

We will invoke the general hypothesis $f \in \mathrm{LNV}_{(x_0,y_0)}(X \times Y) \cup \mathrm{LNV}_{x_0}^{y_0}(X \times Y)$ on many occasions. Let us examine this recurring hypothesis more closely. Is this hypothesis redundant? Can it be simplified in some way? These two simple questions have the same simple answer: No! Surprisingly, neither of the two function classes $\mathrm{LNV}_{(x_0,y_0)}(X \times Y)$ and $\mathrm{LNV}_{x_0}^{y_0}(X \times Y)$ contains the other. The following example constructs a function which establishes that

$$\mathrm{LNV}_{(x_0,y_0)}(X \times Y) \not\subset \mathrm{LNV}_{x_0}^{y_0}(X \times Y).$$

Example 7.11 *Let* $X := Y := \mathbb{R}$ *and* $x_0 := y_0 := 0$, *and note that* 0 *is a limit point of* $\mathbb{R}$. *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$f(x, y) := \chi_{(-1,1)}(x) \cdot \chi_{(-1,1)}(y) \quad \textit{for all} \quad (x, y) \in \mathbb{R}^2,$$

where $\chi_{(-1,1)}$ *denotes the characteristic function of the open interval* $(-1, 1)$. *Since*

$$f(x, y) = 1 \quad \textit{for all} \quad (x, y) \in (-1, 1)^2,$$

it follows that $f \in \mathrm{LNV}_{(0,0)}(\mathbb{R}^2)$. *Since the cross-sections of* f *satisfy* $f_x = 0$ *and* $f^y = 0$ *for all real* x *and* y *such that* $|x| \geq 1$ *and* $|y| \geq 1$, *it follows that* $f \notin \mathrm{LNV}_0^0(\mathbb{R}^2)$.

Because the function in the previous example has the form $f = \chi_{(-1,1)} \otimes \chi_{(-1,1)}$, a straightforward calculation shows that $\Upsilon_{(0,0)} f = f$, which means that f is a fixed-point of the operator $\Upsilon_{(0,0)}$. In general, every rank-one tensor product in $\mathrm{LNV}_{(x_0,y_0)}(X \times Y) \cup \mathrm{LNV}_{x_0}^{y_0}(X \times Y)$ is a fixed-point of the asymptotic splitting operator $\Upsilon_{(x_0,y_0)}$.

Conversely, the following example constructs a function which establishes that

$$\mathrm{LNV}_{x_0}^{y_0}(X \times Y) \not\subset \mathrm{LNV}_{(x_0,y_0)}(X \times Y).$$

Example 7.12 *Let* $X := Y := (0, \infty)$ *and* $x_0 := y_0 := 0$, *and note that* 0 *is a limit point of the interval* $(0, \infty)$. *Consider the function* $f : (0, \infty)^2 \to \mathbb{R}$ *defined by*

$$f(x, y) := x - y \quad \textit{for all} \quad (x, y) \in (0, \infty)^2.$$

Since $f(x, y) = 0$ *on the line* $y = x$, *there is no neighborhood of* $(0, 0)$ *in which* f *is nonzero; consequently,* $f \notin \mathrm{LNV}_{(0,0)}((0, \infty)^2)$. *If* $x > 0$, *the cross-section* f_x *satisfies* $f_x(y) > 0$ *for* *all real y with $0 < y < x$. This shows that $f_x \in \mathrm{LNV}_0(0,\infty)$ for all real $x > 0$. Similarly, $f^y \in \mathrm{LNV}_0(0,\infty)$ for all real $y > 0$. We conclude that $f \in \mathrm{LNV}_0^0\left((0,\infty)^2\right)$.*

Since $f \in \mathrm{LNV}_0^0\left((0,\infty)^2\right)$, we can consider whether the function $\Upsilon_{(0,0)}f$ is well-defined. Let us calculate the following iterated one-dimensional limit at an arbitrary point $(x,y) \in (0,\infty)^2$:

$$\lim_{x'\to 0^+}\left(\lim_{y'\to 0^+}\frac{(x-y')\,(x'-y)}{x'-y'}\right) = x\cdot\lim_{x'\to 0^+}\left(\frac{x'-y}{x'}\right) = x\cdot\left(1-y\cdot\lim_{x'\to 0^+}\frac{1}{x'}\right) = -\infty.$$

Since this iterated one-dimensional limit fails to exist over the real numbers, the rectilinear limit of the same expression as $\{x'\to 0,\ y'\to 0\}$ fails to exist. We conclude that the function $\Upsilon_{(0,0)}f$ is not well-defined.

Now that we have carefully defined the asymptotic splitting operator and finished exploring the main hypotheses of this definition, we are ready to develop the fundamental properties of the operator. This is the focus of the next subsection.

7.2.2 Asymptotic Structure Theory

This subsection develops three major theorems for the asymptotic splitting operator. These theorems have both theoretical and practical value, and shed considerable light by establishing general conditions which ensure that the operator produces a function with a desirable structure.

Recall that the specialized hypotheses of Proposition 7.10 ensure that the function $\Upsilon_{(x_0,y_0)}f$ is well-defined *and* has the desired structure—a tensor product of rank one. The following theorem gives more general hypotheses sufficient to ensure that the function $\Upsilon_{(x_0,y_0)}f$ is both well-defined and a tensor product of rank one:

Theorem 7.13 (Weak Asymptotic Structure) *Let $X, Y \subset \overline{\mathbb{R}}$ be arbitrary subsets, let x_0 and y_0 be limit points of X and Y, respectively, and let*

$$f \in \mathrm{LNV}_{(x_0,y_0)}(X\times Y) \cup \mathrm{LNV}_{x_0}^{y_0}(X\times Y). \tag{7.3}$$

If there exist functions $g \in \mathrm{LNV}_{x_0}(X)$ and $h \in \mathrm{LNV}_{y_0}(Y)$ and a constant $c \in \mathbb{R}\backslash\{0\}$ such *that*

$$\lim_{x' \to x_0} \frac{f(x', y)}{g(x')} = c \cdot h(y) \quad \textit{for all} \quad y \in Y \tag{7.4}$$

$$\lim_{y' \to y_0} \frac{f(x, y')}{h(y')} = c \cdot g(x) \quad \textit{for all} \quad x \in X, \tag{7.5}$$

then the following rectilinear limit exists and satisfies

$$\lim_{\{x' \to x_0,\ y' \to y_0\}} \frac{f(x', y')}{g(x')\, h(y')} = c. \tag{7.6}$$

In addition, the function $\Upsilon_{(x_0,y_0)} f$ *is well-defined and has the following structure:*

$$\Upsilon_{(x_0,y_0)} f = c \cdot (g \otimes h).$$

Proof. Let us divide both sides of equation 7.4 by $h(y)$ and replace y by y' to obtain

$$\lim_{x' \to x_0} \frac{f(x', y')}{g(x')\, h(y')} = c \quad \text{for all} \quad y' \in \text{a-supp}\, h.$$

Since the previous equation asserts that the left-hand side is actually independent of y', and since by hypothesis y_0 is a limit point of a-supp h, we can take the limit of both sides of this equation as $y' \to y_0$ to obtain

$$\lim_{y' \to y_0} \left(\lim_{x' \to x_0} \frac{f(x', y')}{g(x')\, h(y')} \right) = c.$$

Similarly, dividing both sides of equation 7.5 by $g(x)$, replacing x by x', and taking the limit as $x' \to x_0$ yields

$$\lim_{x' \to x_0} \left(\lim_{y' \to y_0} \frac{f(x', y')}{g(x')\, h(y')} \right) = c.$$

By the definition of the rectilinear limit, the previous two equations together imply that equation (7.6) holds.

Since $c \neq 0$, hypothesis (7.3) allows us to take the reciprocal of equation (7.6) to obtain the new rectilinear limit

$$\lim_{\{x' \to x_0,\ y' \to y_0\}} \frac{g(x')\, h(y')}{f(x', y')} = \frac{1}{c}.$$

Since equation (7.4) is independent of y', the following rectilinear limit also holds:

$$\lim_{\{x' \to x_0,\ y' \to y_0\}} \frac{f(x', y)}{g(x')} = c \cdot h(y) \quad \text{for all} \quad y \in Y.$$

Similarly, since equation (7.5) is independent of x', the following rectilinear limit holds as well:

$$\lim_{\{x' \to x_0,\ y' \to y_0\}} \frac{f(x, y')}{h(y')} = c \cdot g(x) \quad \text{for all} \quad x \in X.$$

Multiplying the previous three rectilinear limits together yields

$$\lim_{\{x' \to x_0,\ y' \to y_0\}} \left(\frac{f(x, y')}{h(y')} \frac{f(x', y)}{g(x')} \frac{g(x')\, h(y')}{f(x', y')} \right) = \frac{c \cdot c}{c} \cdot g(x)\, h(y) \quad \text{for all} \quad (x, y) \in X \times Y.$$

After cancellation, we obtain

$$\Upsilon_{(x_0, y_0)}\, f(x, y) := \lim_{\{x' \to x_0,\ y' \to y_0\}} \frac{f(x, y')\, f(x', y)}{f(x', y')} = c \cdot g(x)\, h(y) \quad \text{for all} \quad (x, y) \in X \times Y,$$

as desired. ■

We will use Weak Asymptotic Structure Theorem 7.13 to prove the uniqueness of dual asymptotic expansions in the next section. Although this theorem is useful in a theoretical context, it has one practical drawback: Given a function

$$f \in \mathrm{LNV}_{(x_0, y_0)}(X \times Y) \cup \mathrm{LNV}_{x_0}^{y_0}(X \times Y),$$

it does not tell us how to *find* the functions $g \in \mathrm{LNV}_{x_0}(X)$ and $h \in \mathrm{LNV}_{y_0}(Y)$. Later in this subsection, we will develop Strong Asymptotic Structure Theorem 7.16 to address this drawback in the case $f \in \mathrm{LNV}_{x_0}^{y_0}(X \times Y)$; this theorem will use the asymptotic composition operators $\circ_{x_0}$ and $\circ^{y_0}$ to *generate* the functions g and h, but will do so *under slightly stronger hypotheses*—hence the name of the theorem.

As Example 7.12 demonstrated, the hypothesis $f \in \mathrm{LNV}_{x_0}^{y_0}(X \times Y)$ does not by itself guarantee that the function $\Upsilon_{(x_0, y_0)} f$ is well-defined. Under this hypothesis, we can, however, use the asymptotic composition operators of the previous section to specify a sufficient condition for $\Upsilon_{(x_0, y_0)} f$ to be well-defined. The following theorem states this sufficient condition precisely:

Theorem 7.14 (Iterated Asymptotic Composition) *Let $X, Y \subset \overline{\mathbb{R}}$ be arbitrary subsets, let x_0 and y_0 be limit points of X and Y, respectively, and let $f \in \mathrm{LNV}_{x_0}^{y_0}(X \times Y)$.*

Assume that the asymptotic compositions $f \circ_{x_0} f$ and $f \circ^{y_0} f$ are well-defined and satisfy

$$f \circ_{x_0} f \in \mathrm{LNV}_{y_0}^{y_0}(Y \times Y) \quad \textit{and} \quad f \circ^{y_0} f \in \mathrm{LNV}_{x_0}^{x_0}(X \times X).$$

If the iterated asymptotic compositions

$$(f \circ_{x_0} f) \circ^{y_0} f : X \times Y \to \mathbb{R} \quad \textit{and} \quad (f \circ^{y_0} f) \circ_{x_0} f : X \times Y \to \mathbb{R}$$

are well-defined and are equal as functions, then the function $\Upsilon_{(x_0,y_0)} f$ is also well-defined and satisfies

$$\Upsilon_{(x_0,y_0)} f = (f \circ_{x_0} f) \circ^{y_0} f = (f \circ^{y_0} f) \circ_{x_0} f.$$

Proof. Let $(x, y) \in X \times Y$ denote an arbitrary point all throughout this proof. By the definitions of the asymptotic composition operators,

$$\begin{aligned}
((f \circ_{x_0} f) \circ^{y_0} f)(x,y) &:= \lim_{y' \to y_0} \frac{f(x,y')}{(f \circ_{x_0} f)(y,y')} = \lim_{y' \to y_0} \frac{f(x,y')}{\displaystyle\lim_{x' \to x_0} \frac{f(x',y')}{f(x',y)}} \\
&= \lim_{y' \to y_0} \left(f(x,y') \lim_{x' \to x_0} \frac{f(x',y)}{f(x',y')} \right) = \lim_{y' \to y_0} \left(\lim_{x' \to x_0} \frac{f(x,y')\, f(x',y)}{f(x',y')} \right) \\
((f \circ^{y_0} f) \circ_{x_0} f)(x,y) &:= \lim_{x' \to x_0} \frac{f(x',y)}{(f \circ^{y_0} f)(x',x)} = \lim_{x' \to x_0} \frac{f(x',y)}{\displaystyle\lim_{y' \to y_0} \frac{f(x',y')}{f(x,y')}} \\
&= \lim_{x' \to x_0} \left(f(x',y) \lim_{y' \to y_0} \frac{f(x,y')}{f(x',y')} \right) = \lim_{x' \to x_0} \left(\lim_{y' \to y_0} \frac{f(x,y')\, f(x',y)}{f(x',y')} \right).
\end{aligned}$$

Since all of the asymptotic compositions and all of the ensuing quotients in our analysis are well-defined by hypothesis, all of the limits in our analysis exist. Furthermore, the hypothesis

$$((f \circ_{x_0} f) \circ^{y_0} f)(x,y) = ((f \circ^{y_0} f) \circ_{x_0} f)(x,y)$$

and the above analysis together imply that

$$\lim_{y' \to y_0} \left(\lim_{x' \to x_0} \frac{f(x,y')\, f(x',y)}{f(x',y')} \right) = \lim_{x' \to x_0} \left(\lim_{y' \to y_0} \frac{f(x,y')\, f(x',y)}{f(x',y')} \right).$$

We conclude that the rectilinear limit

$$\Upsilon_{(x_0,y_0)} f(x,y) := \lim_{\{x' \to x_0,\ y' \to y_0\}} \frac{f(x,y')\, f(x',y)}{f(x',y')}$$

exists, and its value is by definition the common value of the two iterated one-dimensional limits displayed above. By the preceding analysis, this common value is

$$\Upsilon_{(x_0,y_0)} f(x,y) = ((f \circ_{x_0} f) \circ^{y_0} f)(x,y) = ((f \circ^{y_0} f) \circ_{x_0} f)(x,y),$$

which yields the desired result. ■

If we impose additional hypotheses on the structure of the asymptotic compositions $f \circ_{x_0} f$ and $f \circ^{y_0} f$, the previous theorem will allow us to deduce the structure of the function $\Upsilon_{(x_0,y_0)} f$. The following definition provides the terminology and notation we need to describe these additional hypotheses:

Definition 7.15 *Let S be an arbitrary set and let $f : S \to \mathbb{R}$ be an arbitrary function. If $f(s) \neq 0$ for all $s \in S$, we say that f* ***is globally nonvanishing****. We denote the set of all such real-valued functions on S by* $\mathrm{GNV}(S)$.

Using the notation of the previous definition, we can now classify every real-valued function on X according to its place in the following hierarchy:

$$\mathrm{GNV}(X) \subset \mathrm{LNV}_{x_0}(X) \subset \mathrm{SNV}_{x_0}(X) \subset \mathbb{R}^X.$$

Similarly, we can classify every real-valued function on $X \times Y$ according to its place in the following hierarchy:

$$\begin{aligned} \mathrm{GNV}(X \times Y) \ &\subset\ \mathrm{LNV}_{(x_0,y_0)}(X \times Y) \cap \mathrm{LNV}_{x_0}^{y_0}(X \times Y) \subset \mathrm{LNV}_{x_0}^{y_0}(X \times Y) \\ &\subset\ \mathrm{LNV}_{(x_0,y_0)}(X \times Y) \cup \mathrm{LNV}_{x_0}^{y_0}(X \times Y) \subset \mathbb{R}^{X \times Y}. \end{aligned}$$

Earlier in this subsection, we used the function classes $\mathrm{LNV}_{x_0}(X)$ and $\mathrm{LNV}_{y_0}(Y)$ and the hypothesis $f \in \mathrm{LNV}_{(x_0,y_0)}(X \times Y) \cup \mathrm{LNV}_{x_0}^{y_0}(X \times Y)$ to develop Weak Asymptotic Structure Theorem 7.13, which gave a sufficient condition for the function $\Upsilon_{(x_0,y_0)} f$ to be well-defined and have the structure of a tensor product of *rank exactly one*. We will now use the proper subclasses $\mathrm{GNV}(X)$ and $\mathrm{GNV}(Y)$ and the slightly stronger hypothesis $f \in \mathrm{LNV}_{x_0}^{y_0}(X \times Y)$ to develop Strong Asymptotic Structure Theorem 7.16, which gives a sufficient condition for the function $\Upsilon_{(x_0,y_0)}f$ to be well-defined and have the structure of a tensor product of *rank at most one*; this theorem also includes a converse:

Theorem 7.16 (Strong Asymptotic Structure) *Let $X, Y \subset \overline{\mathbb{R}}$ be arbitrary subsets, let x_0 and y_0 be limit points of X and Y, respectively, and let $f \in \mathrm{LNV}_{x_0}^{y_0}(X \times Y)$. If there exist functions $g \in \mathrm{GNV}(X)$ and $h \in \mathrm{GNV}(Y)$ such that*

$$(f \circ_{x_0} f)(y, y') = \frac{h(y')}{h(y)} \qquad \text{for all} \qquad (y, y') \in Y^2 \tag{7.7a}$$

$$(f \circ^{y_0} f)(x', x) = \frac{g(x')}{g(x)} \qquad \text{for all} \qquad (x', x) \in X^2 \tag{7.7b}$$

and a constant $c \in \mathbb{R}$ such that

$$\lim_{\{x' \to x_0,\ y' \to y_0\}} \frac{f(x', y')}{g(x')\, h(y')} = c, \tag{7.8}$$

then the following two identities hold:

$$\lim_{x' \to x_0} \frac{f(x', y)}{g(x')} = c \cdot h(y) \quad \text{for all} \quad y \in Y \tag{7.9a}$$

$$\lim_{y' \to y_0} \frac{f(x, y')}{h(y')} = c \cdot g(x) \quad \text{for all} \quad x \in X. \tag{7.9b}$$

In addition, the function $\Upsilon_{(x_0,y_0)}f$ is well-defined and has the following structure:

$$\Upsilon_{(x_0,y_0)}f = c \cdot (g \otimes h). \tag{7.10}$$

Conversely, if $c \neq 0$ and equation (7.9) holds, then equations (7.7) and (7.8) hold also; by the first part of this theorem, equation (7.10) holds as well.

Proof. Assume that equations (7.7) and (7.8) hold. We will show that equation (7.9) must hold also.

In order to establish equation (7.9a), we first invoke the definition of the asymptotic composition operator $\circ_{x_0}$ to rewrite equation (7.7a) as

$$\lim_{x' \to x_0} \frac{f(x', y')}{f(x', y)} = \frac{h(y')}{h(y)} \quad \text{for all} \quad (y, y') \in Y^2.$$

We can solve this equation for $h(y)$ to obtain

$$h(y) = h(y') \lim_{x' \to x_0} \frac{f(x', y)}{f(x', y')} = \lim_{x' \to x_0} \frac{h(y')\, f(x', y)}{f(x', y')} \quad \text{for all} \quad (y, y') \in Y^2.$$

Since $h(y)$ is independent of y', the right-hand side of this equation is also independent of y'; therefore, we can take the limit of this equation as $y' \to y_0$ to obtain

$$h(y) = \lim_{y' \to y_0} \left(\lim_{x' \to x_0} \frac{h(y')\, f(x', y)}{f(x', y')} \right) \quad \text{for all} \quad y \in Y.$$

By the definition of the rectilinear limit, equation (7.8) yields

$$c = \lim_{y' \to y_0} \left(\lim_{x' \to x_0} \frac{f(x', y')}{g(x')\, h(y')} \right).$$

Multiplying the previous two equations together yields

$$\begin{aligned} c \cdot h(y) &= \lim_{y' \to y_0} \left(\lim_{x' \to x_0} \frac{f(x', y')}{g(x')\, h(y')} \right) \cdot \lim_{y' \to y_0} \left(\lim_{x' \to x_0} \frac{h(y')\, f(x', y)}{f(x', y')} \right) \\ &= \lim_{y' \to y_0} \left(\lim_{x' \to x_0} \frac{f(x', y')}{g(x')\, h(y')} \frac{h(y')\, f(x', y)}{f(x', y')} \right) \\ &= \lim_{y' \to y_0} \left(\lim_{x' \to x_0} \frac{f(x', y)}{g(x')} \right) = \lim_{x' \to x_0} \frac{f(x', y)}{g(x')} \quad \text{for all} \quad y \in Y. \end{aligned}$$

This establishes that the limit in equation (7.9a) exists and has the desired value.

Similarly, we can establish equation (7.9b) by using equation (7.7b) to derive the identity

$$g(x) = \lim_{x' \to x_0} \left(\lim_{y' \to y_0} \frac{g(x')\, f(x, y')}{f(x', y')} \right) \quad \text{for all} \quad x \in X$$

and equation (7.8) to obtain

$$c = \lim_{x' \to x_0} \left(\lim_{y' \to y_0} \frac{f(x', y')}{g(x')\, h(y')} \right).$$

Multiplying the previous two equations together yields

$$c \cdot g(x) = \lim_{x' \to x_0} \left(\lim_{y' \to y_0} \frac{f(x', y')}{g(x')\, h(y')} \frac{g(x')\, f(x, y')}{f(x', y')} \right) = \lim_{y' \to y_0} \frac{f(x, y')}{h(y')} \quad \text{for all} \quad x \in X.$$

This establishes that the limit in equation (7.9b) exists and has the desired value.

We will now prove that equation (7.10) holds by invoking Iterated Asymptotic Composition Theorem 7.14; note that we *cannot* simply invoke Weak Asymptotic Structure Theorem 7.13 since we must allow for the possibility that $c = 0$. Hypothesis (7.7) implies that the asymptotic compositions $f \circ_{x_0} f$ and $f \circ^{y_0} f$ are well-defined and satisfy

$$f \circ_{x_0} f \in \mathrm{GNV}\left(Y^2\right) \quad \text{and} \quad f \circ^{y_0} f \in \mathrm{GNV}\left(X^2\right).$$

This in turn allows us to consider whether the iterated asymptotic compositions

$$(f \circ_{x_0} f) \circ^{y_0} f : X \times Y \to \mathbb{R} \quad \text{and} \quad (f \circ^{y_0} f) \circ_{x_0} f : X \times Y \to \mathbb{R}$$

are well-defined.

Let $(x, y) \in X \times Y$ denote an arbitrary point. The definitions of the asymptotic composition operators $\circ_{x_0}$ and $\circ^{y_0}$ and hypothesis (7.7) together imply

$$\begin{aligned}
((f \circ_{x_0} f) \circ^{y_0} f)(x, y) &:= \lim_{y' \to y_0} \frac{f(x, y')}{(f \circ_{x_0} f)(y, y')} = \lim_{y' \to y_0} \frac{f(x, y')}{h(y')/h(y)} \\
&= h(y) \lim_{y' \to y_0} \frac{f(x, y')}{h(y')} \qquad (7.11a) \\
((f \circ^{y_0} f) \circ_{x_0} f)(x, y) &:= \lim_{x' \to x_0} \frac{f(x', y)}{(f \circ^{y_0} f)(x', x)} = \lim_{x' \to x_0} \frac{f(x', y)}{g(x')/g(x)} \\
&= g(x) \lim_{x' \to x_0} \frac{f(x', y)}{g(x')}. \qquad (7.11b)
\end{aligned}$$

Substituting equation (7.9) into equation (7.11) yields

$$\begin{aligned}
((f \circ_{x_0} f) \circ^{y_0} f)(x, y) &= h(y)\,(c \cdot g(x)) = c \cdot (g \otimes h)(x, y) \\
((f \circ^{y_0} f) \circ_{x_0} f)(x, y) &= g(x)\,(c \cdot h(y)) = c \cdot (g \otimes h)(x, y).
\end{aligned}$$

This shows that the iterated asymptotic compositions $(f \circ_{x_0} f) \circ^{y_0} f$ and $(f \circ^{y_0} f) \circ_{x_0} f$ are well-defined and satisfy

$$(f \circ_{x_0} f) \circ^{y_0} f = c \cdot (g \otimes h) = (f \circ^{y_0} f) \circ_{x_0} f.$$

By Iterated Asymptotic Composition Theorem 7.14, the function $\Upsilon_{(x_0,y_0)}f$ is also well-defined and satisfies

$$\Upsilon_{(x_0,y_0)}f = c \cdot (g \otimes h).$$

To establish the converse, let $c \in \mathbb{R}\backslash\{0\}$ and assume that equation (7.9) holds. We must show that equations (7.7) and (7.8) hold also. To show that equation (7.7a) holds, we let $(y, y') \in Y^2$ denote an arbitrary point and appeal to the definition of the covariant asymptotic composition operator:

$$\begin{aligned}(f \circ_{x_0} f)(y, y') &:= \lim_{x' \to x_0} \frac{f(x', y')}{f(x', y)} = \lim_{x' \to x_0} \left(\frac{f(x', y')}{g(x')} \frac{g(x')}{f(x', y)} \right) \\ &= \lim_{x' \to x_0} \left(\frac{f(x', y')}{g(x')} \right) \cdot \lim_{x' \to x_0} \left(\frac{g(x')}{f(x', y)} \right) = c \cdot h(y') \, \frac{1}{c \cdot h(y)} = \frac{h(y')}{h(y)}.\end{aligned}$$

Similarly, to show that equation (7.7b) holds, we let $(x', x) \in X^2$ denote an arbitrary point and appeal to the definition of the contravariant asymptotic composition operator:

$$(f \circ^{y_0} f)(x', x) := \lim_{y' \to y_0} \frac{f(x', y')}{f(x, y')} = \lim_{y' \to y_0} \left(\frac{f(x', y')}{h(y')} \right) \cdot \lim_{y' \to y_0} \left(\frac{h(y')}{f(x, y')} \right) = \frac{g(x')}{g(x)}.$$

Since $c \neq 0$, equation (7.8) holds by Weak Asymptotic Structure Theorem 7.13. ∎

We will use Strong Asymptotic Structure Theorem 7.16 to prove a necessary and sufficient condition for the existence of dual asymptotic expansions in the next section. It is the *two-way implication* contained in Strong Asymptotic Structure Theorem 7.16 that will render this existence condition *both necessary and sufficient*.

Note that Strong Asymptotic Structure Theorem 7.16 can also be used to prove Weak Asymptotic Structure Theorem 7.13—under stronger hypotheses. That is precisely why we proved Weak Asymptotic Structure Theorem 7.13 directly, under weaker hypotheses; this in turn allowed us to use Weak Asymptotic Structure Theorem 7.13 to prove the converse portion of Strong Asymptotic Structure Theorem 7.16.

Now that we have finished developing the necessary theory, we will find it helpful to consider some illustrative examples. This is the focus of the next subsection.

7.2.3 Meditations upon Illuminating Examples

It is illuminating to examine how various equations in the statement of Strong Asymptotic Structure Theorem 7.16 behave when the functions involved are scaled by nonzero constants. Assume that equations (7.7) and (7.8) hold for the functions given, and let $\alpha, \beta, \gamma \in \mathbb{R}\backslash\{0\}$. If we scale the functions f and g and h by the constants α and β and γ, respectively, equation (7.7) still holds since

$$\begin{aligned}(\alpha f \circ_{x_0} \alpha f)(y, y') &= (f \circ_{x_0} f)(y, y') = \frac{h(y')}{h(y)} = \frac{\gamma h(y')}{\gamma h(y)} \\ (\alpha f \circ^{y_0} \alpha f)(x', x) &= (f \circ^{y_0} f)(x', x) = \frac{g(x')}{g(x)} = \frac{\beta g(x')}{\beta g(x)}.\end{aligned}$$

The invariance of equation (7.7) under scaling means that equation (7.7) cannot determine the functions g and h uniquely beyond a multiplicative constant. Equation (7.8) compensates for this lack of uniqueness by calculating the coefficient c in a manner that takes scaling into account:

$$\lim_{\{x' \to x_0,\ y' \to y_0\}} \frac{\alpha f(x', y')}{\beta g(x')\, \gamma h(y')} = \frac{\alpha}{\beta\gamma} \cdot \left(\lim_{\{x' \to x_0,\ y' \to y_0\}} \frac{f(x', y')}{g(x')\, h(y')} \right) = \frac{\alpha}{\beta\gamma}\, c.$$

Under this scaling of functions, equation (7.10) yields

$$\Upsilon_{(x_0, y_0)} \alpha f = \frac{\alpha}{\beta\gamma}\, c \cdot (\beta g \otimes \gamma h) = \alpha\, c \cdot (g \otimes h) = \alpha \Upsilon_{(x_0, y_0)} f,$$

which is the expected result since the asymptotic splitting operator is homogeneous with respect to nonzero constants.

The following example builds on Example 7.12, and demonstrates that *equation (7.8) of Strong Asymptotic Structure Theorem 7.16 cannot be derived from equation (7.7)—they are independent hypotheses:*

Example 7.17 *Let* $X := Y := (0, \infty)$ *and* $x_0 := y_0 := 0$*, and note that* 0 *is a limit point of the interval* $(0, \infty)$*. Consider the function* $f : (0, \infty)^2 \to \mathbb{R}$ *defined by*

$$f(x, y) := x - y \quad \textit{for all} \quad (x, y) \in (0, \infty)^2.$$

We showed in Example 7.12 that

$$f \in \mathrm{LNV}_0^0\left((0, \infty)^2\right) \setminus \mathrm{LNV}_{(0,0)}\left((0, \infty)^2\right).$$

We subsequently showed that the function $\Upsilon_{(0,0)} f$ *is not well-defined. Nevertheless, the* *asymptotic compositions* $f \circ_0 f$ *and* $f \circ^0 f$ *are both well-defined, and satisfy*

$$(f \circ_0 f)(y, y') \;:=\; \lim_{x' \to 0^+} \frac{x' - y'}{x' - y} = \frac{y'}{y} \quad \textit{for all} \quad (y, y') \in (0, \infty)^2$$

$$(f \circ^0 f)(x', x) \;:=\; \lim_{y' \to 0^+} \frac{x' - y'}{x - y'} = \frac{x'}{x} \quad \textit{for all} \quad (x', x) \in (0, \infty)^2.$$

We conclude that equation (7.7) of Strong Asymptotic Structure Theorem 7.16 holds with $g(x) := x$ *and* $h(y) := y$*, which satisfy* $g, h \in \mathrm{GNV}(0, \infty)$.

Now calculate the following one-dimensional limit for arbitrary real $y' > 0$*:*

$$\lim_{x' \to 0^+} \frac{f(x', y')}{g(x')\, h(y')} = \lim_{x' \to 0^+} \frac{x' - y'}{x'\, y'} = \lim_{x' \to 0^+} \left(\frac{1}{y'} - \frac{1}{x'} \right) = \frac{1}{y'} - \lim_{x' \to 0^+} \frac{1}{x'} = -\infty.$$

Since this one-dimensional limit fails to produce a well-defined real-valued function of y' *in any deleted neighborhood of* 0*, the iterated one-dimensional limit*

$$\lim_{y' \to 0^+} \left(\lim_{x' \to 0^+} \frac{f(x', y')}{g(x')\, h(y')} \right)$$

is ill-posed, and the rectilinear limit

$$\lim_{\{x' \to 0^+,\ y' \to 0^+\}} \frac{f(x', y')}{g(x')\, h(y')}$$

therefore does not exist. We conclude that equation (7.8) of Strong Asymptotic Structure Theorem 7.16 does not hold.

It is tempting to think that the rectilinear limit in equation (7.8) failed to exist in the previous example because $f \notin \mathrm{LNV}_{(0,0)}\left((0, \infty)^2\right)$. Readers who find themselves sorely tempted in this fashion should retreat to a monastery[1] or convent[2] *immediately*, for this disquieting thought is pernicious and untrue! Such readers are urged to seek inner clarity through the quiet contemplation of the two functions defined for all

[1] After seven consecutive years of graduate school, the author is thoroughly acquainted with the monastic lifestyle, and employs this metaphor with the deepest reverence and respect for the life that is devoted to the pursuit and attainment of higher truths.

[2] After seven consecutive years of graduate school, the author still knows regrettably little about life in a convent. Perhaps a good postdoctoral fellowship will remedy this!

$(x, y) \in (0, \infty)^2$ by

$$f_0(x, y) := (x - y) \exp\left(\frac{1}{x + y}\right) \quad \text{and} \quad f_1(x, y) := \left(\frac{x - y}{x + y}\right)^2.$$

Like the function f in the previous example, both of these functions satisfy

$$f_i \in \mathrm{LNV}_0^0\left((0, \infty)^2\right) \setminus \mathrm{LNV}_{(0,0)}\left((0, \infty)^2\right) \quad \text{for} \quad i = 0, 1.$$

Unlike the function f in the previous example, equations (7.7) and (7.8) of Strong Asymptotic Structure Theorem 7.16 both hold for f_0 and f_1, and equation (7.10) yields

$$\Upsilon_{(0,0)} f_0(x, y) = 0 \cdot \left(x \exp\frac{1}{x} \cdot y \exp\frac{1}{y}\right) = 0 \quad \text{and} \quad \Upsilon_{(0,0)} f_1(x, y) = 1 \cdot (1 \cdot 1) = 1.$$

The contemplation of these mysteries will surely bring the troubled reader inner peace, for that is the *essential* purpose of this *singular* practice.[3]

From our meditations, we emerge renewed, comforted by the knowledge that the two tensor products $\Upsilon_{(0,0)} f_i$ satisfy

$$\operatorname{rank} \Upsilon_{(0,0)} f_i = i \quad \text{for} \quad i = 0, 1.$$

This illuminating fact reveals to us that the asymptotic splitting operator can generate tensor products of either rank zero or rank one. In the mundane world, we are far more interested in the rank-one case. The result $\Upsilon_{(0,0)} f_0 = 0$ in the rank-zero case simply indicates that the function f_0 has a subtle and ethereal nature that is not well-suited for asymptotic analysis via the operator $\Upsilon_{(0,0)}$ at the point $(0, 0)$; this does not preclude the possibility that f_0 may be amenable to analysis via the operator $\Upsilon_{(x_0,y_0)}$ at *another* point $(x_0, y_0) \neq (0, 0)$!

We constructed the functions f and f_0 and f_1 just discussed in order to illustrate various points concerning the hypotheses and conclusions of Strong Asymptotic Structure Theorem 7.16. The following more typical example builds on Example 7.7 from the previous section, and illustrates how Strong Asymptotic Structure Theorem 7.16 is generally applied:

Example 7.18 *Let $X := Y := (0, \frac{\pi}{2})$, and note that $x_0 := 0$ and $y_0 := \frac{\pi}{2}$ are limits points*

[3] Hint hint! Please read *between* the lines. (Squinting may prove helpful.)

of the interval $(0, \frac{\pi}{2})$. *Consider the function* $f : (0, \frac{\pi}{2})^2 \to \mathbb{R}$ *defined by*

$$f(x,y) := \sin(x+y) \quad \textit{for all} \quad (x,y) \in \left(0, \frac{\pi}{2}\right)^2,$$

and note that $f \in \mathrm{GNV}\left((0, \frac{\pi}{2})^2\right)$. *We showed in Example 7.7 that the asymptotic compositions* $f \circ_0 f$ *and* $f \circ^{\pi/2} f$ *are well-defined and satisfy*

$$\begin{aligned}
(f \circ_0 f)(y, y') &= \frac{\sin y'}{\sin y} &&\textit{for all} &&(y, y') \in \left(0, \frac{\pi}{2}\right)^2 \\
(f \circ^{\pi/2} f)(x', x) &= \frac{\cos x'}{\cos x} &&\textit{for all} &&(x', x) \in \left(0, \frac{\pi}{2}\right)^2.
\end{aligned}$$

We conclude that equation (7.7) of Strong Asymptotic Structure Theorem 7.16 holds with $g(x) := \cos x$ *and* $h(y) := \sin y$, *which satisfy* $g, h \in \mathrm{GNV}(0, \frac{\pi}{2})$.

Now calculate the following iterated one-dimensional limits:

$$\begin{aligned}
\lim_{x' \to 0^+} \left(\lim_{y' \to \pi/2^-} \frac{f(x', y')}{g(x')\, h(y')} \right) &= \lim_{x' \to 0^+} \left(\lim_{y' \to \pi/2^-} \frac{\sin(x' + y')}{\cos x' \, \sin y'} \right) \\
&= \lim_{x' \to 0^+} \frac{\cos x'}{\cos x'} = \lim_{x' \to 0^+} 1 = 1 \\
\lim_{y' \to \pi/2^-} \left(\lim_{x' \to 0^+} \frac{f(x', y')}{g(x')\, h(y')} \right) &= \lim_{y' \to \pi/2^-} \left(\lim_{x' \to 0^+} \frac{\sin(x' + y')}{\cos x' \, \sin y'} \right) \\
&= \lim_{y' \to \pi/2^-} \frac{\sin y'}{\sin y'} = \lim_{y' \to \pi/2^-} 1 = 1.
\end{aligned}$$

Since these two iterated one-dimensional limits both exist and are equal in value, the rectilinear limit of the specified quotient as $\{x' \to 0^+,\ y' \to \frac{\pi}{2}^-\}$ *exists, and equation (7.8) of Strong Asymptotic Structure Theorem 7.16 therefore holds with constant* $c = 1$. *Invoking the theorem, we conclude that the function* $\Upsilon_{(0,\pi/2)} f$ *is well-defined and has the structure* $\Upsilon_{(0,\pi/2)} f(x,y) = \cos x \, \sin y$.

Of course, we can also obtain the result of the previous example by applying the definition of the asymptotic splitting operator directly. This example, in addition to illustrating the use of Strong Asymptotic Structure Theorem 7.16, also illustrates another point worth noting: We recognize the function $\Upsilon_{(0,\pi/2)} f(x,y) = \cos x \, \sin y$ as the *first* term of the familiar addition formula

$$\sin(x+y) \equiv \cos x \, \sin y + \sin x \, \cos y.$$

This is not an accident! The addition formula for $\sin(x+y)$ is actually a dual asymptotic expansion as $x \to 0^+$ or $y \to \frac{\pi}{2}^-$. In the next section, we will learn how to generate *all* the terms of a dual asymptotic expansion as $x \to x_0$ or $y \to y_0$ using a simple iterative algorithm based on the asymptotic splitting operator $\Upsilon_{(x_0,y_0)}$.

7.3 Dual Asymptotic Expansions

In this section, we will formally introduce dual asymptotic expansions and explore the nature of their duality. After this, we will use the asymptotic splitting operator and Weak Asymptotic Structure Theorem 7.13 to develop a *uniqueness theorem for dual asymptotic expansions*. We will then convert this uniqueness theorem into an *iterative algorithm* which uses the asymptotic splitting operator to *generate* the unique dual asymptotic expansion of a given function at a given point. After this, we will use the asymptotic composition operators and Strong Asymptotic Structure Theorem 7.16 to develop a *necessary and sufficient condition for the existence of dual asymptotic expansions* under slightly stronger hypotheses. We will conclude this section by converting this existence theorem into an *iterative algorithm* which *decides* the question of existence by using the asymptotic composition operators to *generate* the unique dual asymptotic expansion of a given function at a given point whenever the expansion exists.

7.3.1 Definitions and Fundamental Results

This subsection formally defines dual asymptotic expansions and gives a simple classification scheme based on two sets of supplementary hypotheses—one set of hypotheses is weak, and one set of hypotheses is strong. This classification will help us to develop various aspects of the subsequent theory under the most appropriate set of hypotheses for a particular purpose.

Let $X, Y \subset \overline{\mathbb{R}}$ be arbitrary subsets, and let x_0 and y_0 be limit points of X and Y, respectively. Let $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}\backslash\{0\}$ and $\{g_i\}_{i=0}^{n-1} \subset \mathbb{R}^X$ and $\{h_i\}_{i=0}^{n-1} \subset \mathbb{R}^Y$, where n is either a positive integer or ∞. Assume that $\{g_i(x)\}_{i=0}^{n-1}$ is an asymptotic sequence as $x \to x_0$ and that $\{h_i(y)\}_{i=0}^{n-1}$ is an asymptotic sequence as $y \to y_0$.

Since the sum

$$\sum_{i=0}^{n-1} c_i \cdot g_i(x)\, h_i(y) \tag{7.12}$$

is both an asymptotic series as $x \to x_0$ for each fixed $y \in Y$ and an asymptotic series as $y \to y_0$ for each fixed $x \in X$, we call this sum a **dual asymptotic series as $x \to x_0$ or $y \to y_0$**. If $n < \infty$, the resulting finite sum is a tensor product. If $n = \infty$, we treat the resulting infinite sum as a formal series. In either case, we denote the partial sums of this series by

$$s_m(x,y) := \sum_{i=0}^{m-1} c_i \cdot g_i(x)\, h_i(y) \quad \text{for} \quad 0 \le m < n+1,$$

where $s_0 := 0$ by convention.

Remark 7.19 *Under the modest additional hypotheses $\{g_i\}_{i=0}^{n-1} \subset \mathrm{SNV}_{x_0}(X)$ and $\{h_i\}_{i=0}^{n-1} \subset \mathrm{SNV}_{y_0}(Y)$, which are more than satisfied in typical applications, the partial sum s_m becomes a particular kind of tensor product expression which we studied extensively in Chapter 4: Since Theorem 6.15 implies that the asymptotic sequences $\{g_i\}_{i=0}^{n-1}$ and $\{h_i\}_{i=0}^{n-1}$ are linearly independent sets, the partial sum s_m is a tensor product expression in binormal form! Binormal Form Minimality Theorem 4.38 further implies that the tensor product s_m has rank m; thus, s_m has the smallest possible number of terms among all equivalent tensor product representations.*

Let $f : X \times Y \to \mathbb{R}$ be an arbitrary function. We define the remainders of f with respect to the dual asymptotic series (7.12) by

$$r_m := f - s_m \quad \text{for} \quad 0 \le m < n+1.$$

In particular, $r_0 := f$. We are now ready to make the following formal definition:

Definition 7.20 *Under the hypotheses above, we say that the dual asymptotic series (7.12) is a **dual asymptotic expansion of** $f(x,y)$ **to** n **terms as** $x \to x_0$ **or** $y \to y_0$ if both of the following univariate asymptotic expansions hold:*

$$f^y(x) \sim \sum_{i=0}^{n-1} (c_i \cdot h_i(y)) \cdot g_i(x) \quad \textit{as} \quad x \to x_0 \quad \textit{for all} \quad y \in Y \tag{7.13a}$$

$$f_x(y) \sim \sum_{i=0}^{n-1} (c_i \cdot g_i(x)) \cdot h_i(y) \quad \textit{as} \quad y \to y_0 \quad \textit{for all} \quad x \in X. \tag{7.13b}$$

We express this by writing

$$f(x,y) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x)\, h_i(y) \quad \textit{as} \quad x \to x_0 \quad \textit{or} \quad y \to y_0. \tag{7.14}$$

We also call this series a ***dual asymptotic expansion of*** f ***to*** n ***terms at*** (x_0, y_0).

The following remark states an important convention concerning dual asymptotic expansions:

Remark 7.21 *Whenever we assert that dual asymptotic expansion (7.14) holds, it is understood that all of the coefficients* $\{c_i\}_{i=0}^{n-1}$ *are nonzero. Unlike univariate asymptotic expansions, dual asymptotic expansions do not contain any zero terms—every term makes an actual contribution!*

We adopt the convention of the previous remark because the algorithms we use to generate dual asymptotic expansions never produce trivial terms. The following definition formalizes the hypotheses we will need to ensure that these algorithms work properly:

Definition 7.22 *We say that dual asymptotic expansion (7.14) is a* ***weak dual asymptotic expansion*** *if the asymptotic sequences satisfy*

$$\{g_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{x_0}(X) \quad \textit{and} \quad \{h_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{y_0}(Y)$$

and the remainders satisfy

$$\{r_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{(x_0,y_0)}(X \times Y) \cup \mathrm{LNV}_{x_0}^{y_0}(X \times Y).$$

We denote the set of all real-valued functions on $X \times Y$ *which have weak dual asymptotic expansions to* n *terms at* (x_0, y_0) *by* $\mathrm{WDAE}^n_{(x_0,y_0)}(X \times Y)$. *Since* $r_0 := f$, *it follows that*

$$\mathrm{WDAE}^n_{(x_0,y_0)}(X \times Y) \subset \mathrm{LNV}_{(x_0,y_0)}(X \times Y) \cup \mathrm{LNV}_{x_0}^{y_0}(X \times Y).$$

We say that dual asymptotic expansion (7.14) is a ***strong dual asymptotic expansion*** *if the asymptotic sequences satisfy*

$$\{g_i\}_{i=0}^{n-1} \subset \mathrm{GNV}(X) \quad \textit{and} \quad \{h_i\}_{i=0}^{n-1} \subset \mathrm{GNV}(Y)$$

and the remainders satisfy

$$\{r_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{x_0}^{y_0}(X \times Y).$$

We denote the set of all real-valued functions on $X \times Y$ *which have strong dual asymptotic expansions to* n *terms at* (x_0, y_0) *by* $\mathrm{SDAE}^n_{(x_0,y_0)}(X \times Y)$*. Since every strong dual asymptotic expansion is also a weak dual asymptotic expansion, and since* $r_0 := f$*, it follows that*

$$\mathrm{SDAE}^n_{(x_0,y_0)}(X \times Y) \subset \mathrm{WDAE}^n_{(x_0,y_0)}(X \times Y) \cap \mathrm{LNV}_{x_0}^{y_0}(X \times Y).$$

The following characterization theorem gives a simple criterion for determining whether a *given* formal series is a weak dual asymptotic expansion of a given function; note that this criterion does not *generate* the series expansion, however:

Theorem 7.23 (Characterization) *Let* $X, Y \subset \overline{\mathbb{R}}$ *be arbitrary subsets, and let* x_0 *and* y_0 *be limit points of* X *and* Y*, respectively. Assume that*

$$f \in \mathrm{LNV}_{(x_0,y_0)}(X \times Y) \cup \mathrm{LNV}_{x_0}^{y_0}(X \times Y).$$

Let $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}\backslash\{0\}$ *and* $\{g_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{x_0}(X)$ *and* $\{h_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{y_0}(Y)$*, where* n *is either a positive integer or* ∞*. Assume that the remainders* $\{r_i\}_{i=0}^{n-1}$ *of the function* f *with respect to the formal series*

$$\sum_{i=0}^{n-1} c_i \cdot (g_i \otimes h_i)$$

satisfy

$$\{r_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{(x_0,y_0)}(X \times Y) \cup \mathrm{LNV}_{x_0}^{y_0}(X \times Y). \tag{7.15}$$

Under these hypotheses, we can assert that $\{g_i(x)\}_{i=0}^{n-1}$ *is an asymptotic sequence as* $x \to x_0$*, that* $\{h_i(y)\}_{i=0}^{n-1}$ *is an asymptotic sequence as* $y \to y_0$*, and that the following weak dual asymptotic expansion holds*

$$f(x, y) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x)\, h_i(y) \quad \textit{as} \quad x \to x_0 \quad \textit{or} \quad y \to y_0 \tag{7.16}$$

if and only if the following two identities hold for $0 \leq i < n$*:*

$$\lim_{x \to x_0} \frac{r_i(x,y)}{g_i(x)} = c_i \cdot h_i(y) \quad \textit{for all} \quad y \in Y \tag{7.17a}$$

$$\lim_{y \to y_0} \frac{r_i(x,y)}{h_i(y)} = c_i \cdot g_i(x) \quad \textit{for all} \quad x \in X. \tag{7.17b}$$

Proof. If we assume that $\{g_i(x)\}_{i=0}^{n-1}$ is an asymptotic sequence as $x \to x_0$, that $\{h_i(y)\}_{i=0}^{n-1}$ is an asymptotic sequence as $y \to y_0$, and that weak dual asymptotic expansion (7.16) holds, Proposition 6.17 for univariate asymptotic expansions implies that equation (7.17) holds for $0 \leq i < n$. Proving the converse of this statement is more involved.

Assume that equation (7.17) holds for $0 \leq i < n$. Since $r_{i+1} = r_i - c_i \cdot (g_i \otimes h_i)$, we can rewrite equation (7.17) to obtain the following identities, which also hold for $0 \leq i < n$:

$$\lim_{x \to x_0} \frac{r_{i+1}(x,y)}{g_i(x)} = 0 \quad \text{for all} \quad y \in Y \tag{7.18a}$$

$$\lim_{y \to y_0} \frac{r_{i+1}(x,y)}{h_i(y)} = 0 \quad \text{for all} \quad x \in X. \tag{7.18b}$$

Since $\{g_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{x_0}(X)$ and $\{h_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{y_0}(Y)$, and since the remainders $\{r_i\}_{i=0}^{n-1}$ satisfy hypothesis (7.15), there exist constants $\{\xi_i\}_{i=0}^{n-1} \subset X$ and $\{\eta_i\}_{i=0}^{n-1} \subset Y$ with the following properties for $0 \leq i < n$:

$$\begin{aligned} g_i(\xi_i) &\neq 0 \quad \text{and} \quad r_i(\xi_i, y) \neq 0 \quad \text{as} \quad y \to y_0 \\ h_i(\eta_i) &\neq 0 \quad \text{and} \quad r_i(x, \eta_i) \neq 0 \quad \text{as} \quad x \to x_0. \end{aligned}$$

As a result, the following algebraic manipulations are justified, and the values of the limits follow from equations (7.17a) and (7.18a):

$$\begin{aligned} \lim_{x \to x_0} \frac{g_{i+1}(x)}{g_i(x)} &= \lim_{x \to x_0} \left(\frac{g_{i+1}(x)}{r_{i+1}(x,\eta_{i+1})} \frac{r_{i+1}(x,\eta_{i+1})}{g_i(x)} \right) \\ &= \lim_{x \to x_0} \frac{g_{i+1}(x)}{r_{i+1}(x,\eta_{i+1})} \cdot \lim_{x \to x_0} \frac{r_{i+1}(x,\eta_{i+1})}{g_i(x)} \\ &= \frac{1}{c_{i+1} \cdot h_{i+1}(\eta_{i+1})} \cdot 0 = 0. \end{aligned}$$

Since the above limit holds for $0 \leq i < n-1$, it follows that $\{g_i(x)\}_{i=0}^{n-1}$ is an asymptotic sequence as $x \to x_0$.

Similarly, the following algebraic manipulations are justified, and the values of the limits follow from equations (7.17b) and (7.18b):

$$\lim_{y \to y_0} \frac{h_{i+1}(y)}{h_i(y)} = \lim_{y \to y_0} \frac{h_{i+1}(y)}{r_{i+1}(\xi_{i+1}, y)} \cdot \lim_{y \to y_0} \frac{r_{i+1}(\xi_{i+1}, y)}{h_i(y)} = \frac{1}{c_{i+1} \cdot g_{i+1}(\xi_{i+1})} \cdot 0 = 0.$$

Since the above limit holds for $0 \leq i < n-1$, it follows that $\{h_i(y)\}_{i=0}^{n-1}$ is an asymptotic sequence as $y \to y_0$. Furthermore, since we assumed that equation (7.17) holds for $0 \leq i < n$, Proposition 6.17 implies that weak dual asymptotic expansion (7.16) holds. ■

Now that we have formally introduced dual asymptotic expansions, given a basic classification scheme, and developed a simple characterization theorem, we are ready to explore dual asymptotic expansions in greater depth. The next subsection delves deeply into the nature of the duality in dual asymptotic expansions.

7.3.2 A Rigorous Exploration of Duality

What is the nature of the duality in a dual asymptotic expansion? One kind of duality that is clearly evident in both Definition 7.20 and Characterization Theorem 7.23 is the duality between *variables* and *parameters*. In equations (7.13a) and (7.17a), the variable x approaches x_0 while the parameter y remains fixed, whereas in equations (7.13b) and (7.17b), the variable y approaches y_0 while the parameter x remains fixed.

Another kind of duality that we observe in these equations is the duality between *basis functions* and *coefficient functions*. If we assume that $\{g_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{x_0}(X)$ and $\{h_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{y_0}(Y)$, Theorem 6.15 implies that the asymptotic sequences $\{g_i\}_{i=0}^{n-1}$ and $\{h_i\}_{i=0}^{n-1}$ are linearly independent sets in the function spaces $\mathbb{R}^X$ and $\mathbb{R}^Y$, respectively; this means that each of these two sets is a basis of its own $\mathbb{R}$-linear span. In equations (7.13a) and (7.17a), the elements of $\{g_i\}_{i=0}^{n-1}$ serve as basis functions while the elements of $\{c_i \cdot h_i\}_{i=0}^{n-1}$ serve as coefficient functions. Conversely, in equations (7.13b) and (7.17b), the elements of $\{h_i\}_{i=0}^{n-1}$ serve as basis functions while the elements of $\{c_i \cdot g_i\}_{i=0}^{n-1}$ serve as coefficient functions. We can make this kind of duality even more apparent by invoking the formalism of *dual bases*, as follows:

Assume that $n < \infty$. Let U denote the set of all functions $u \in \mathbb{R}^X$ such that $u(x)$ has an asymptotic expansion to n terms with respect to the asymptotic sequence $\{g_i(x)\}_{i=0}^{n-1}$ as $x \to x_0$. Similarly, let V denote the set of all functions $v \in \mathbb{R}^Y$ such that $v(y)$ has an asymptotic expansion to n terms with respect to the asymptotic sequence $\{h_i(y)\}_{i=0}^{n-1}$ as $y \longrightarrow y_0$. The sets U and V are subspaces of the function spaces $\mathbb{R}^X$ and $\mathbb{R}^Y$, respectively.

Define the linear functionals $\phi_i \in U^*$ for $i = 0, 1, \ldots, n-1$ by the recursive formula

$$\phi_i(u) := \lim_{x \to x_0} \left[\left(u(x) - \sum_{j=0}^{i-1} \phi_j(u) \cdot g_j(x) \right) \cdot \frac{1}{g_i(x)} \right] \quad \text{for all} \quad u \in U.$$

For each $u \in U$, the linear functionals $\{\phi_i\}_{i=0}^{n-1}$ calculate the coefficients in the asymptotic expansion

$$u(x) \sim \sum_{i=0}^{n-1} \phi_i(u) \cdot g_i(x) \quad \text{as} \quad x \longrightarrow x_0.$$

Since this asymptotic expansion exists by definition of the function space U, Proposition 6.17 ensures that the linear functionals $\{\phi_i\}_{i=0}^{n-1}$ are well-defined. Similarly, the linear functionals $\psi_i \in V^*$ defined for $i = 0, 1, \ldots, n-1$ by the recursive formula

$$\psi_i(v) := \lim_{y \to y_0} \left[\left(v(y) - \sum_{j=0}^{i-1} \psi_j(v) \cdot h_j(y) \right) \cdot \frac{1}{h_i(y)} \right] \quad \text{for all} \quad v \in V$$

calculate the coefficients in the asymptotic expansion

$$v(y) \sim \sum_{i=0}^{n-1} \psi_i(v) \cdot h_i(y) \quad \text{as} \quad y \longrightarrow y_0$$

for each $v \in V$, and are well-defined by Proposition 6.17.

Let W denote the set of all functions $f \in \mathbb{R}^{X \times Y}$ such that

$$f^y \in U \quad \text{for all} \quad y \in Y \qquad \text{and} \qquad f_x \in V \quad \text{for all} \quad x \in X$$

and such that

$$(x \mapsto \psi_i(f_x)) \in U \quad \text{and} \quad (y \mapsto \phi_i(f^y)) \in V \quad \text{for} \quad 0 \le i \le n-1.$$

Since U and V are function spaces and the maps $f \mapsto f_x$ and $f \mapsto f^y$ are linear, it follows that the set W is a subspace of the function space $\mathbb{R}^{X \times Y}$. The function space W consists of all real-valued functions on $X \times Y$ whose cross-sections have univariate asymptotic expansions to n terms with respect to the asymptotic sequences $\{g_i\}_{i=0}^{n-1}$ and $\{h_i\}_{i=0}^{n-1}$, subject to the proviso that the resulting coefficient functions *also* have univariate asymptotic expansions to n terms with respect to the asymptotic sequences $\{g_i\}_{i=0}^{n-1}$ and $\{h_i\}_{i=0}^{n-1}$. In particular, $U \otimes V \subset W$.

For $i = 0, 1, \ldots, n-1$, define the parametric extension $\Phi_i \in \mathrm{Hom}_{\mathbb{R}}(W, V)$ of the linear functional $\phi_i \in U^*$ by

$$(\Phi_i f)(y) := \phi_i(f^y) \quad \text{for all} \quad f \in W \quad \text{and} \quad y \in Y.$$

Similarly, for $i = 0, 1, \ldots, n-1$, define the parametric extension $\Psi_i \in \mathrm{Hom}_{\mathbb{R}}(W, U)$ of the linear functional $\psi_i \in V^*$ by

$$(\Psi_i f)(x) := \psi_i(f_x) \quad \text{for all} \quad f \in W \quad \text{and} \quad x \in X.$$

The definition of the function space W ensures that these parametric extensions are well-defined.

In addition, define the n-dimensional function spaces $U_n := \mathrm{span}_{\mathbb{R}}\{g_i\}_{i=0}^{n-1}$ and $V_n := \mathrm{span}_{\mathbb{R}}\{h_i\}_{i=0}^{n-1}$. Since every finite asymptotic series is an asymptotic expansion of itself, the subspace inclusions $U_n \subset U$ and $V_n \subset V$ hold. Since the coefficients in a finite asymptotic series are identical to the coefficients in the corresponding asymptotic expansion, it follows that

$$\phi_i(g_j) = \psi_i(h_j) = \delta_{ij} \quad \text{for} \quad 0 \leq i, j \leq n-1,$$

where δ_{ij} denotes the Kronecker delta. We conclude that *restricting* the linear functionals $\{\phi_i\}_{i=0}^{n-1}$ to the subspace U_n yields the basis $\{\phi_i|U_n\}_{i=0}^{n-1}$ of U_n^* which is *dual* to the basis $\{g_i\}_{i=0}^{n-1}$ of U_n. Similarly, *restricting* the linear functionals $\{\psi_i\}_{i=0}^{n-1}$ to the subspace V_n yields the basis $\{\psi_i|V_n\}_{i=0}^{n-1}$ of V_n^* which is *dual* to the basis $\{h_i\}_{i=0}^{n-1}$ of V_n.

In order to clarify the role of these dual bases in the context of dual asymptotic expansions, let us now change to a new notation based on *extensions* rather than *restrictions*. Denote the dual bases by

$$g_i^* := \phi_i|U_n \quad \text{and} \quad h_i^* := \psi_i|V_n \quad \text{for} \quad 0 \leq i \leq n-1.$$

Denote the *extensions* of the dual bases from U_n to U and V_n to V by

$$\tilde{g}_i^* := \phi_i \quad \text{and} \quad \tilde{h}_i^* := \psi_i \quad \text{for} \quad 0 \leq i \leq n-1.$$

Finally, denote the *parametric extensions* of these linear functions from U to W and V to

W by

$$G_i^* := \Phi_i \quad \text{and} \quad H_i^* := \Psi_i \quad \text{for} \quad 0 \le i \le n-1.$$

Now assume that the function $f : X \times Y \to \mathbb{R}$ has the dual asymptotic expansion

$$f(x,y) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x)\, h_i(y) \quad \text{as} \quad x \to x_0 \quad \text{or} \quad y \to y_0.$$

The inclusions $U_n \subset U$ and $V_n \subset V$ imply that $f \in W$. Proposition 6.17 further implies that

$$G_i^* f = c_i \cdot h_i \quad \text{and} \quad H_i^* f = c_i \cdot g_i \quad \text{for} \quad 0 \le i \le n-1.$$

Remark 7.24 *These equations clearly express the nature of the duality between the basis functions and the coefficient functions in a dual asymptotic expansion of f to n terms at (x_0, y_0): The basis functions $\{g_i\}_{i=0}^{n-1} \subset U_n$ generate a dual basis $\{g_i^*\}_{i=0}^{n-1} \subset U_n^*$ whose elements extend to linear operators $\{G_i^*\}_{i=0}^{n-1}$ on the function space W; when applied to the function $f \in W$, these linear operators generate the coefficient functions $\{c_i \cdot h_i\}_{i=0}^{n-1}$. Conversely, the basis functions $\{h_i\}_{i=0}^{n-1} \subset V_n$ generate a dual basis $\{h_i^*\}_{i=0}^{n-1} \subset V_n^*$ whose elements extend to linear operators $\{H_i^*\}_{i=0}^{n-1}$ on the function space W; when applied to the function $f \in W$, these linear operators generate the coefficient functions $\{c_i \cdot g_i\}_{i=0}^{n-1}$.*

Here is a brief but interesting aside: If the constant function $1 \in U \cap V$, then

$$\mathbb{R} \subset U \subset W \supset V \supset \mathbb{R}.$$

These inclusions ensure that the compositions $\Phi_i \Psi_j$ and $\Psi_j \Phi_i$ are well-defined, and allow us to consider whether the parametric extensions Φ_i and Ψ_j *commute* for $0 \le i, j \le n-1$. The following example settles this question definitively:

Example 7.25 *Let $X := Y := (0, \infty)$, and let $x_0 := 0$ and $y_0 := \infty$. Note that 0 and ∞ are limit points of the open interval $(0, \infty)$. Let $n := 1$, and define the constant functions $g_0 := h_0 := 1$. Let the function spaces U, V, and W be defined as above, and note that $1 \in U \cap V$ since $U_0 := V_0 := \mathbb{R}$. Define the function $f : (0,\infty)^2 \to \mathbb{R}$ by*

$$f(x,y) := e^{-xy} \quad \textit{for all} \quad (x,y) \in (0,\infty)^2.$$

For any real $y > 0$, we obtain

$$f^y(x) \sim 1 \quad \textit{as} \quad x \to 0^+,$$

which implies that $\Phi_0 f = 1$ and hence that $\Psi_0\Phi_0 f = 1$. For any real $x > 0$, we obtain

$$f_x(y) \sim 0 \quad \textit{as} \quad y \to \infty,$$

which implies that $\Psi_0 f = 0$ and hence that $\Phi_0\Psi_0 f = 0$. In summary,

$$\Psi_0\Phi_0 f = 1 \neq 0 = \Phi_0\Psi_0 f,$$

which means that the parametric extensions Φ_0 and Ψ_0 do not commute.[4]

Now that we understand the nature of the duality in dual asymptotic expansions, let us turn our attention to the fundamental questions of uniqueness and existence. These questions are the focus of the last two subsections of this chapter.

7.3.3 Uniqueness Theory for Weak Dual Asymptotic Expansions

In this subsection, we will determine conditions sufficient to ensure the uniqueness of dual asymptotic expansions under the weak hypotheses, assuming existence. We begin with the following standard example, which illustrates an important difference between *univariate* asymptotic expansions and *dual* asymptotic expansions in regard to the question of uniqueness:

Example 7.26 *Let $X := (0, \infty)$ and note that $x_0 := \infty$ is a limit point of X. The univariate expression e^{-x} has asymptotic expansions with respect to both of the asymptotic sequences $\{x^{-n}\}_{n=0}^{\infty} \subset \mathrm{GNV}(0, \infty)$ and $\{e^{-nx}\}_{n=0}^{\infty} \subset \mathrm{GNV}(0, \infty)$ as $x \to \infty$:*

$$\begin{aligned} e^{-x} &\sim \sum_{n=0}^{\infty} 0 \cdot x^{-n} \quad \textit{as} \quad x \to \infty \\ e^{-x} &\sim 0 + e^{-x} + \sum_{n=2}^{\infty} 0 \cdot e^{-nx} \quad \textit{as} \quad x \to \infty. \end{aligned}$$

We conclude that in the univariate case, a function on X which has an asymptotic expansion at x_0 does *not* uniquely determine the corresponding asymptotic sequence $\{g_i\}_{i=0}^{n-1}$.

[4] We can construct a similar example using *finite* limit points $x_0 := y_0 := 0$ of a *bounded* interval $X := Y := (0, 1)$ in conjunction with the function

$$f(x, y) := \frac{(1 - x)\sin y}{x + y} \quad \text{for all} \quad (x, y) \in (0, 1)^2.$$

In fact, selecting an asymptotic sequence that yields a meaningful asymptotic expansion for a given univariate function is a nontrivial undertaking that requires either excellent human judgement or a sophisticated computer algorithm.

The bivariate case is actually *simpler:* If a function on $X \times Y$ has a *weak dual asymptotic expansion* to n terms at (x_0, y_0), then the corresponding univariate asymptotic sequences $\{g_i\}_{i=0}^{n-1}$ and $\{h_i\}_{i=0}^{n-1}$ are *uniquely determined* up to nonzero multiplicative constants! We will establish this property as part of the following *uniqueness theorem for weak dual asymptotic expansions:*

Theorem 7.27 (Uniqueness) *Let $X, Y \subset \overline{\mathbb{R}}$ be arbitrary subsets, and let x_0 and y_0 be limit points of X and Y, respectively. If $f \in \mathrm{WDAE}^n_{(x_0,y_0)}(X \times Y)$ has the following weak dual asymptotic expansion*

$$f(x,y) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x)\, h_i(y) \quad \textit{as} \quad x \to x_0 \quad \textit{or} \quad y \to y_0$$

with remainders $\{r_i\}_{i=0}^{n}$, then the following rectilinear limits exists and satisfy

$$\lim_{\{x \to x_0,\ y \to y_0\}} \frac{r_i(x,y)}{g_i(x)\, h_i(y)} = c_i \quad \textit{for} \quad 0 \le i < n, \tag{7.19}$$

and the functions $\{\Upsilon_{(x_0,y_0)}\, r_i\}_{i=0}^{n-1}$ are well-defined and satisfy

$$\Upsilon_{(x_0,y_0)}\, r_i = c_i \cdot (g_i \otimes h_i) \quad \textit{for} \quad 0 \le i < n. \tag{7.20}$$

Furthermore, under these hypotheses, the terms $\{c_i \cdot (g_i \otimes h_i)\}_{i=0}^{n-1}$ of a dual asymptotic expansion of the function f at (x_0, y_0) are uniquely determined, the univariate functions $\{g_i\}_{i=0}^{n-1}$ and $\{h_i\}_{i=0}^{n-1}$ are uniquely determined up to nonzero multiplicative constants, and the coefficients $\{c_i\}_{i=0}^{n-1}$ with respect to a particular choice of functions $\{g_i\}_{i=0}^{n-1}$ and $\{h_i\}_{i=0}^{n-1}$ are uniquely determined as well.

Proof. Characterization Theorem 7.23 implies that the following two identities hold for $0 \le i < n$:

$$\begin{aligned} \lim_{x \to x_0} \frac{r_i(x,y)}{g_i(x)} &= c_i \cdot h_i(y) \quad \text{for all} \quad y \in Y \\ \lim_{y \to y_0} \frac{r_i(x,y)}{h_i(y)} &= c_i \cdot g_i(x) \quad \text{for all} \quad x \in X. \end{aligned}$$

By a direct application of Weak Asymptotic Structure Theorem 7.13, the rectilinear limits in equation (7.19) exist and take on the values specified by the coefficients $\{c_i\}_{i=0}^{n-1}$, and the functions $\{\Upsilon_{(x_0,y_0)}\, r_i\}_{i=0}^{n-1}$ are well-defined and satisfy equation (7.20).

To show the uniqueness of the terms, suppose that f has another weak dual asymptotic expansion to n terms at (x_0, y_0):

$$f(x,y) \sim \sum_{i=0}^{n-1} \bar{c}_i \cdot \bar{g}_i(x)\, \bar{h}_i(y) \quad \text{as} \quad x \to x_0 \quad \text{or} \quad y \to y_0.$$

By the argument above, the functions $\{\Upsilon_{(x_0,y_0)}\, \bar{r}_i\}_{i=0}^{n-1}$ are well-defined and satisfy

$$\Upsilon_{(x_0,y_0)}\, \bar{r}_i = \bar{c}_i \cdot (\bar{g}_i \otimes \bar{h}_i) \quad \text{for} \quad 0 \le i < n.$$

The uniqueness of the terms will follow easily from the result

$$r_i = \bar{r}_i \quad \text{for} \quad 0 \le i < n, \tag{7.21}$$

which we now prove by induction on i (finite induction if $n < \infty$ and infinite induction if $n = \infty$): When $i = 0$, we obtain $r_0 := f =: \bar{r}_0$. Suppose that $r_i = \bar{r}_i$ for some i satisfying $0 \le i < n - 1$. It follows that

$$c_i \cdot (g_i \otimes h_i) = \Upsilon_{(x_0,y_0)}\, r_i = \Upsilon_{(x_0,y_0)}\, \bar{r}_i = \bar{c}_i \cdot (\bar{g}_i \otimes \bar{h}_i),$$

and thus

$$r_{i+1} = r_i - c_i \cdot (g_i \otimes h_i) = \bar{r}_i - \bar{c}_i \cdot (\bar{g}_i \otimes \bar{h}_i) = \bar{r}_{i+1}.$$

Since $r_0 = \bar{r}_0$, and since $r_i = \bar{r}_i$ implies $r_{i+1} = \bar{r}_{i+1}$ whenever $0 \le i < n - 1$, it follows by induction on i that $r_i = \bar{r}_i$ for all i satisfying $0 \le i < n$.

Applying the operator $\Upsilon_{(x_0,y_0)}$ to equation (7.21) yields the following identity, which holds for $0 \le i < n$:

$$c_i \cdot g_i(x)\, h_i(y) = \bar{c}_i \cdot \bar{g}_i(x)\, \bar{h}_i(y) \quad \text{for all} \quad (x,y) \in X \times Y. \tag{7.22}$$

This identity asserts that all of the *terms* of a dual asymptotic expansion of the function f at (x_0, y_0) are uniquely determined. We will now show that each *function* appearing in each term is uniquely determined up to a nonzero multiplicative constant:

For the rest of this proof, let i denote any natural number satisfying $0 \leq i < n$. Since $g_i \in \mathrm{LNV}_{x_0}(X)$ and $h_i \in \mathrm{LNV}_{y_0}(Y)$, there exist $\xi_i \in X$ and $\eta_i \in Y$ such that $g_i(\xi_i) \neq 0$ and $h_i(\eta_i) \neq 0$. Since $c_i \neq 0$ and $\bar{c}_i \neq 0$ by assumption, identity (7.22) implies that

$$c_i \cdot g_i(\xi_i)\, h_i(\eta_i) = \bar{c}_i \cdot \bar{g}_i(\xi_i)\, \bar{h}_i(\eta_i) \neq 0,$$

which further implies that $\bar{g}_i(\xi_i) \neq 0$ and $\bar{h}_i(\eta_i) \neq 0$ as well. Substituting $y = \eta_i$ in identity (7.22) and solving for $g_i(x)$ yields

$$g_i(x) = \frac{\bar{c}_i \cdot \bar{h}_i(\eta_i)}{c_i \cdot h_i(\eta_i)} \cdot \bar{g}_i(x) \quad \text{for all} \quad x \in X,$$

as desired. Similarly, substituting $x = \xi_i$ in identity (7.22) and solving for $h_i(y)$ yields

$$h_i(y) = \frac{\bar{c}_i \cdot \bar{g}_i(\xi_i)}{c_i \cdot g_i(\xi_i)} \cdot \bar{h}_i(y) \quad \text{for all} \quad y \in Y,$$

as desired.

To show that the coefficients are unique *with respect to a particular choice of univariate functions*, assume that $g_i = \bar{g}_i$ and $h_i = \bar{h}_i$. Since $r_i = \bar{r}_i$ as well, equation (7.19) implies

$$c_i = \lim_{\{x \to x_0,\ y \to y_0\}} \frac{r_i(x,y)}{g_i(x)\, h_i(y)} = \lim_{\{x \to x_0,\ y \to y_0\}} \frac{\bar{r}_i(x,y)}{\bar{g}_i(x)\, \bar{h}_i(y)} = \bar{c}_i,$$

which completes the proof. ■

What does this uniqueness theorem really mean? The following remark states the central implication:

Remark 7.28 *Given a function* $f \in \mathrm{WDAE}^n_{(x_0,y_0)}(X \times Y)$, *Uniqueness Theorem 7.27 asserts that the weak dual asymptotic expansion of* f *to* n *terms at* (x_0, y_0) *is uniquely determined by the function* f *and the point* (x_0, y_0) *alone!*

Given a function $f \in \mathrm{WDAE}^n_{(x_0,y_0)}(X \times Y)$, the following iterative algorithm uses the asymptotic splitting operator $\Upsilon_{(x_0,y_0)}$ to *generate* the weak dual asymptotic expansion of f to n terms at (x_0, y_0). This algorithm is a slight modification of Grid Interpolation Algorithm 5.4; the chief differences are that we always use the *same* splitting point (x_0, y_0) instead of selecting *different* splitting points (x_i, y_i) for $0 \leq i < n$, and we use the *general case* of the asymptotic splitting operator rather than a *special case*. Uniqueness Theorem 7.27 proves that this algorithm is correct:

Algorithm 7.29 (Weak Dual Asymptotic Expansion) *Let $f : X \times Y \to \mathbb{R}$, and assume that $1 \leq n < \infty$. If $f \in \mathrm{WDAE}^{\,n}_{(x_0,y_0)}(X \times Y)$, then this algorithm is guaranteed to return the unique weak dual asymptotic expansion of f to n terms at (x_0, y_0). By contraposition, if one of the steps below causes the algorithm to abort with an error, then we can conclude that $f \notin \mathrm{WDAE}^{\,n}_{(x_0,y_0)}(X \times Y)$.*

1. *Let $s_0 := 0$ and $r_0 := f$.*
2. *Iterate the following three steps for $i := 0, 1, \ldots, n-1$:*
   (a) *If the function $\Upsilon_{(x_0,y_0)}\, r_i$ is not well-defined, or if $\Upsilon_{(x_0,y_0)}\, r_i = 0$, then abort the algorithm with an error.*
   (b) *Calculate the next partial sum by $s_{i+1} := s_i + \Upsilon_{(x_0,y_0)}\, r_i$.*
   (c) *Calculate the next remainder by $r_{i+1} := f - s_{i+1}$.*
3. *Return the final partial sum s_n.*

Although a proposition and its *contrapositive* are always logically equivalent, the same cannot be said of a proposition and its *converse*. Using this principle of logic as an exegetical tool leads us to a clear understanding of what the previous algorithm does *not* do:

Remark 7.30 *Although Weak Dual Asymptotic Expansion Algorithm 7.29 will run to completion whenever the input satisfies $f \in \mathrm{WDAE}^{\,n}_{(x_0,y_0)}(X \times Y)$, the successful completion of the algorithm does not guarantee that $f \in \mathrm{WDAE}^{\,n}_{(x_0,y_0)}(X \times Y)$. In general, all we can conclude from the successful completion of the algorithm is that the final partial sum s_n is the only possible candidate for the weak dual asymptotic expansion of f to n terms at (x_0, y_0) in the event that $f \in \mathrm{WDAE}^{\,n}_{(x_0,y_0)}(X \times Y)$.*

It is sometimes convenient to eliminate the partial sums $\{s_i\}_{i=0}^n$ from the previous algorithm and formulate the algorithm exclusively in terms of the remainders $\{r_i\}_{i=0}^n$. We can achieve this by substituting the equation $s_{i+1} := s_i + \Upsilon_{(x_0,y_0)}\, r_i$ into the equation $r_{i+1} := f - s_{i+1}$ to obtain

$$r_{i+1} := f - (s_i + \Upsilon_{(x_0,y_0)}\, r_i) = r_i - \Upsilon_{(x_0,y_0)}\, r_i = (I - \Upsilon_{(x_0,y_0)})\, r_i.$$

Since $r_0 := f$, iterating the previous equation yields

$$r_i = (I - \Upsilon_{(x_0,y_0)})^i f \quad \text{for} \quad 0 \leq i \leq n. \tag{7.23}$$

After calculating the final remainder r_n in this way, we can easily recover the final partial sum s_n via $s_n = f - r_n$.

Remark 7.31 *In the terminology of dynamical systems, equation (7.23) tells us that the sequence of remainders* $\{r_i\}_{i=0}^{n}$ *is the orbit of the function* f *under the remainder operator* $I - \Upsilon_{(x_0,y_0)}$. *This point of view further explains why the weak dual asymptotic expansion*

$$s_n = f - r_n = [I - (I - \Upsilon_{(x_0,y_0)})^n]f$$

is uniquely determined by the function f *and the point* (x_0, y_0) *alone whenever* $f \in \mathrm{WDAE}^{\,n}_{(x_0,y_0)}(X \times Y)$.

This concludes our development of the uniqueness theory for dual asymptotic expansions under the weak hypotheses. In the next subsection, we will develop the existence theory for dual asymptotic expansions under the strong hypotheses.

7.3.4 Existence Theory for Strong Dual Asymptotic Expansions

We proved Uniqueness Theorem 7.27 under the *assumption* that a given function has a *weak* dual asymptotic expansion. How do we know when a weak dual asymptotic expansion exists? In order to develop a criterion that is both definitive and practical, we need to impose slightly stronger hypotheses. Keeping in mind that every *strong* dual asymptotic expansion is also a *weak* dual asymptotic expansion, we now state and prove the following *necessary and sufficient condition for the existence of strong dual asymptotic expansions:*

Theorem 7.32 (Existence) *Let* $X, Y \subset \overline{\mathbb{R}}$ *be arbitrary subsets, and let* x_0 *and* y_0 *be limit points of* X *and* Y*, respectively. Assume that* $f \in \mathrm{LNV}_{x_0}^{y_0}(X \times Y)$. *Let* $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}\backslash\{0\}$ *and* $\{g_i\}_{i=0}^{n-1} \subset \mathrm{GNV}(X)$ *and* $\{h_i\}_{i=0}^{n-1} \subset \mathrm{GNV}(Y)$*, where* n *is either a positive integer or* ∞. *Assume that the remainders* $\{r_i\}_{i=0}^{n-1}$ *of the function* f *with respect to the formal series*

$$\sum_{i=0}^{n-1} c_i \cdot (g_i \otimes h_i)$$

satisfy

$$\{r_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{x_0}^{y_0}(X \times Y).$$

Under these hypotheses, we can assert that $\{g_i(x)\}_{i=0}^{n-1}$ *is an asymptotic sequence as* $x \to x_0$*, that* $\{h_i(y)\}_{i=0}^{n-1}$ *is an asymptotic sequence as* $y \to y_0$*, and that the following strong dual* *asymptotic expansion holds*

$$f(x,y) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x)\, h_i(y) \quad \textit{as} \quad x \to x_0 \quad \textit{or} \quad y \to y_0 \tag{7.24}$$

if and only if we can assert for $0 \leq i < n$ *that the following two asymptotic compositions are well-defined and satisfy*

$$(r_i \circ_{x_0} r_i)(y, y') = \frac{h_i(y')}{h_i(y)} \qquad \textit{for all} \qquad (y, y') \in Y^2 \tag{7.25a}$$

$$(r_i \circ^{y_0} r_i)(x', x) = \frac{g_i(x')}{g_i(x)} \qquad \textit{for all} \qquad (x', x) \in X^2, \tag{7.25b}$$

and that the following rectilinear limit exists and satisfies

$$\lim_{\{x \to x_0,\ y \to y_0\}} \frac{r_i(x,y)}{g_i(x)\, h_i(y)} = c_i. \tag{7.26}$$

Please note that when a strong dual asymptotic expansion of f *to* n *terms at* (x_0, y_0) *does exist, its terms are uniquely determined by Uniqueness Theorem 7.27.*

Proof. According to Characterization Theorem 7.23, we can assert that $\{g_i(x)\}_{i=0}^{n-1}$ is an asymptotic sequence as $x \to x_0$, that $\{h_i(y)\}_{i=0}^{n-1}$ is an asymptotic sequence as $y \to y_0$, and that strong dual asymptotic expansion (7.24) holds if and only if the following two identities hold for $0 \leq i < n$:

$$\begin{aligned} \lim_{x \to x_0} \frac{r_i(x,y)}{g_i(x)} &= c_i \cdot h_i(y) \quad \text{for all} \quad y \in Y \\ \lim_{y \to y_0} \frac{r_i(x,y)}{h_i(y)} &= c_i \cdot g_i(x) \quad \text{for all} \quad x \in X. \end{aligned}$$

According to Strong Asymptotic Structure Theorem 7.16, the two identities above hold for $0 \leq i < n$ if and only if equations (7.25) and (7.26) hold for $0 \leq i < n$. WHAM! This completes the proof![5] ■

The previous theorem provides us with considerable insight into the nature of strong dual asymptotic expansions—insight that has both theoretical and practical value. Since

[5] Perhaps the author should have warned the reader that this proof would make a blazingly fast run down the drag strip of the thesis highway—followed by a sudden stop!

the theorem specifies a condition that is *both necessary and sufficient for existence*, the result is definitive. Since the theorem provides a way to actually *generate* the strong dual asymptotic expansion using only the given function f and the point (x_0, y_0), the result is also practical. In fact, the theoretical and practical qualities of this theorem combine to yield a *decision algorithm* for determining whether a given function $f \in \mathrm{LNV}_{x_0}^{y_0}(X \times Y)$ is a member of the function class $\mathrm{SDAE}^n_{(x_0,y_0)}(X \times Y)$. If indeed $f \in \mathrm{SDAE}^n_{(x_0,y_0)}(X \times Y)$, this decision algorithm will *also* do all of the following:

1. The algorithm will *generate* the relevant univariate functions $\{g_i\}_{i=0}^{n-1} \subset \mathrm{GNV}(X)$ and $\{h_i\}_{i=0}^{n-1} \subset \mathrm{GNV}(Y)$.

2. The algorithm will *certify implicitly* that $\{g_i(x)\}_{i=0}^{n-1}$ is an asymptotic sequence as $x \to x_0$ and that $\{h_i(y)\}_{i=0}^{n-1}$ is an asymptotic sequence as $y \to y_0$.

3. The algorithm will *calculate* the coefficients $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}\backslash\{0\}$.

4. The algorithm will *certify implicitly* that the following strong dual asymptotic expansion holds:

$$f(x, y) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x)\, h_i(y) \quad \text{as} \quad x \to x_0 \quad \text{or} \quad y \to y_0.$$

We now give the full specification of this decision algorithm. Existence Theorem 7.32 proves that this algorithm is correct:

Algorithm 7.33 (Strong Dual Asymptotic Expansion) *Let* $f \in \mathrm{LNV}_{x_0}^{y_0}(X \times Y)$ *and assume that* $1 \leq n < \infty$. *If any one of the steps below cannot be completed and therefore causes the algorithm to abort with an error, we can conclude that* $f \notin \mathrm{SDAE}^n_{(x_0,y_0)}(X \times Y)$. *If all of the steps below can be completed and the algorithm terminates normally, we can conclude that* $f \in \mathrm{SDAE}^n_{(x_0,y_0)}(X \times Y)$; *in that case, the algorithm will return the unique strong dual asymptotic expansion of* f *to* n *terms at* (x_0, y_0).

1. *Let* $s_0 := 0$ *and* $r_0 := f$.

2. *Iterate the following six steps for* $i := 0, 1, \ldots, n-1$:

   (a) *If the remainder does not satisfy* $r_i \in \mathrm{LNV}_{x_0}^{y_0}(X \times Y)$, *then abort the algorithm with an error.*

(b) *Calculate the following covariant asymptotic composition:*

$$(r_i \circ_{x_0} r_i)(y, y') = \frac{h_i(y')}{h_i(y)} \quad \textit{for all} \quad (y, y') \in Y^2.$$

If the composition $r_i \circ_{x_0} r_i$ *is not well-defined, or does not have the form specified above for some* $h_i \in \mathrm{GNV}(Y)$, *then abort the algorithm with an error.*

(c) *Calculate the following contravariant asymptotic composition:*

$$(r_i \circ^{y_0} r_i)(x', x) = \frac{g_i(x')}{g_i(x)} \quad \textit{for all} \quad (x', x) \in X^2.$$

If the composition $r_i \circ^{y_0} r_i$ *is not well-defined, or does not have the form specified above for some* $g_i \in \mathrm{GNV}(X)$, *then abort the algorithm with an error.*

(d) *Use the functions* g_i *and* h_i *determined above to calculate the following rectilinear limit:*

$$\lim_{\{x \to x_0,\ y \to y_0\}} \frac{r_i(x, y)}{g_i(x)\, h_i(y)} = c_i.$$

If the rectilinear limit does not exist, or yields the value $c_i = 0$, *then abort the algorithm with an error.*

(e) *Calculate the next partial sum by* $s_{i+1} := s_i + c_i \cdot (g_i \otimes h_i)$.

(f) *Calculate the next remainder by* $r_{i+1} := f - s_{i+1}$.

3. *Return the strong dual asymptotic expansion* s_n.

Can the individual steps of the previous algorithm be carried out in a practical way? The two asymptotic compositions and the rectilinear limit all reduce to ordinary one-dimensional limits of a ratio of functions. For all $i \geq 1$, these limits will produce indeterminate forms such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$, which can usually be evaluated in a straightforward manner by l'Hôpital's rule. Due to the nature of asymptotic analysis, each one-dimensional limit in the i-th iteration will typically require i applications l'Hôpital's rule. For familiar elementary functions, these calculations are totally algorithmic and can often be carried out automatically using a computer algebra system such as Maple.

The step that actually presents the most practical difficulties is the verification of the hypothesis

$$r_i \in \mathrm{LNV}_{x_0}^{y_0}(X \times Y).$$

Even if the original function f is fairly simple, the remainder r_i becomes increasingly complicated as i grows larger. In the next chapter, we will develop a practical approach which eliminates this difficulty—an approach which is intimately related to both l'Hôpital's rule and to the Taylor interpolation properties of dual asymptotic expansions on the rectilinear coordinate lines $x = x_0$ and $y = y_0$.

Chapter 8

Applications to Taylor Interpolation

In this chapter, we will study applications of asymptotic analysis to the Taylor interpolation of real-valued functions of one or two real variables. We will see that asymptotic expansions in one real variable can be characterized by their Taylor interpolation properties at a point on the real line, and that dual asymptotic expansions in two real variables can be characterized by their Taylor interpolation properties on two rectilinear coordinate lines in the plane. In both one and two real variables, we will use the multiplicities of zero points and zero lines to develop simple hypotheses which will have far-reaching consequences for the local algebraic and asymptotic behavior of sequences of functions. These hypotheses will also allow us to exploit l'Hôpital's rule in a systematic way, thereby reducing the evaluation of limits to the computation and evaluation of derivatives. We will use these hypotheses to develop important subclasses of both univariate asymptotic expansions and weak dual asymptotic expansions, and will name these new subclasses after l'Hôpital. In addition, we will develop rigorous remainder theories for Taylor interpolation in both one and two real variables. These remainder theories, which we will develop in both integral and mean-value forms, will yield practical error estimates as well as convergence criteria for l'Hôpital asymptotic expansions with infinitely many terms. We will conclude this chapter by using these two-variable methods to give an original derivation and proof of a well-known uniformly convergent infinite series expansion involving Bessel functions.

8.1 Taylor Interpolation at a Point on the Real Line

We begin our study of Taylor interpolation by first developing the univariate theory. This univariate theory will become the foundation of the bivariate theory that we will build later in this chapter by a systematic application of the process of parametric extension.

8.1.1 Definitions and Fundamental Results

In this subsection, we will develop the notion of multiple zeros for univariate functions and consider its substantial implications for asymptotic analysis in one real variable. Let $X \subset \mathbb{R}$ be an interval, and let $x_0 \in X$. Assume that $f \in C^n(X)$, where $n \in \mathbb{N}$ is arbitrary. We say that x_0 **is a zero of** f **of multiplicity at least** n if

$$D^i f(x_0) = 0 \quad \text{for} \quad 0 \leq i < n.$$

This property is vacuously true whenever $n = 0$. We say that x_0 **is a zero of** f **of multiplicity (exactly)** n if x_0 is a zero of f of multiplicity at least n and

$$D^n f(x_0) \neq 0.$$

In particular, when we assert that x_0 is a zero of f of multiplicity 0, we are saying that $f(x_0) \neq 0$, which means that x_0 is *not* a zero of f. Although this use of terminology may strike the reader as ironic, it is also quite convenient—it saves us from the constant need to assert this as a separate special case. We will therefore invoke this terminology without hesitation. In addition, the author proposes the following convenient terminology:

Definition 8.1 *If x_0 is a zero of f of multiplicity n, we say that f **is point-normalized at** x_0 if*

$$D^n f(x_0) = 1.$$

If x_0 is a zero of f of multiplicity n, we can always divide f by $c := D^n f(x_0) \neq 0$ to obtain a function f/c which is point-normalized at x_0. Point-normalization is useful because it simplifies our formulas. Eliminating the algebraic clutter clarifies the resulting mathematics and allows new insights to emerge in a natural way.

If x_0 is a zero of f of multiplicity n, we can also deduce the local behavior of f and its derivatives up to order n near the point x_0. The following proposition formally states this extremely useful result:

Proposition 8.2 *Let $X \subset \mathbb{R}$ be an interval, and let $x_0 \in X$. Assume that $f \in C^n(X)$, where $n \in \mathbb{N}$ is arbitrary. If x_0 is a zero of f of multiplicity n, then $\{D^i f\}_{i=0}^n \subset \mathrm{LNV}_{x_0}(X)$.*

Proof. First, we will show that $f \in \mathrm{LNV}_{x_0}(X)$. Define the auxiliary function $\hat{f} : X \to \mathbb{R}$ by

$$\hat{f}(x) := \begin{cases} \dfrac{f(x)}{(x - x_0)^n} & \text{if } x \neq x_0, \\ \\ \dfrac{D^n f(x_0)}{n!} & \text{if } x = x_0. \end{cases}$$

Since $f \in C^n(X)$ and $(x \mapsto (x - x_0)^{-n}) \in C^\omega(X \backslash \{x_0\})$, the product of these two functions satisfies $\hat{f} \in C^n(X \backslash \{x_0\})$. Since $\lim_{x \to x_0} \hat{f}(x) = \hat{f}(x_0)$ by n applications of l'Hôpital's rule, it follows that $\hat{f}$ is continuous at x_0. We conclude that $\hat{f} \in C(X)$. In addition, whether $n = 0$ or $n \geq 1$, we obtain the following factorization:

$$f(x) = (x - x_0)^n \cdot \hat{f}(x) \quad \text{for all} \quad x \in X.$$

Since $\hat{f}(x_0) \neq 0$ by hypothesis, and since $\hat{f} \in C(X)$, there exists an open neighborhood N of x_0 such that $\hat{f}(x) \neq 0$ for all $x \in X \cap N$. The factorization above implies that $f(x) \neq 0$ for all $x \in (X \cap N) \backslash \{x_0\}$, which means that $f \in \mathrm{LNV}_{x_0}(X)$.

If $1 \leq i \leq n$, then $D^i f \in C^{n-i}(X)$, and x_0 is a zero of $D^i f$ of multiplicity $n - i$. By the result just proven, $D^i f \in \mathrm{LNV}_{x_0}(X)$, as desired. ■

The following proposition uses the previous result to calculate the limit of the ratio of two functions by repeated applications of l'Hôpital's rule. The proposition also uses the multiplicity of zeros to specify a necessary and sufficient condition for the existence of such limits:

Proposition 8.3 *Let $X \subset \mathbb{R}$ be an interval, and let $x_0 \in X$. Assume that $f, g \in C^n(X)$, where $n \in \mathbb{N}$ is arbitrary. If x_0 is a zero of g of multiplicity n, then the limit of $f(x)/g(x)$ as $x \to x_0$ exists if and only if x_0 is a zero of f of multiplicity at least n, in which case*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{D^n f(x_0)}{D^n g(x_0)}.$$

Proof. Suppose that x_0 is a zero of f of multiplicity at least n. Since $D^n g(x_0) \neq 0$ by assumption, applying l'Hôpital's rule n times to the following limit eliminates the indeterminate form $0/0$ and yields

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{Df(x)}{Dg(x)} = \lim_{x \to x_0} \frac{D^2 f(x)}{D^2 g(x)} = \cdots = \lim_{x \to x_0} \frac{D^n f(x)}{D^n g(x)} = \frac{D^n f(x_0)}{D^n g(x_0)}.$$

Since $\{D^i g\}_{i=0}^n \subset \mathrm{LNV}_{x_0}(X)$ by the previous proposition, all of the quotients above are well-defined as $x \to x_0$. This ensures that our repeated use of l'Hôpital's rule was justified in each case, and allows us to conclude that the original limit exists and takes on the specified value.

Inversely, suppose that x_0 is *not* a zero of f of multiplicity at least n. This implies that x_0 is a zero of f of multiplicity m, where $0 \le m < n$. Since all the hypotheses remain satisfied if f is replaced by $-f$, we may assume without loss of generality that $D^m f(x_0) > 0$. Assume for simplicity that x_0 is *not* the right endpoint of the interval X. This ensures that one-sided limits as $x \to {x_0}^+$ are meaningful. (The case in which x_0 *is* the right endpoint can be handled in a similar fashion by replacing $x \to {x_0}^+$ by $x \to {x_0}^-$, but is omitted for brevity.) Since x_0 is a zero of g of multiplicity $n > m$, we know that $D^m g(x_0) = 0$. Since $D^m g \in \mathrm{LNV}_{x_0}(X)$, there exists $\delta > 0$ such that $D^m g(x) \neq 0$ for all $x \in (x_0, x_0 + \delta)$. Since $D^m g \in C^{n-m}(X)$, continuity implies that $D^m g$ cannot change sign in $(x_0, x_0 + \delta)$. Because all the hypotheses remain satisfied if g is replaced by $-g$, we may assume without loss of generality that $D^m g > 0$ in $(x_0, x_0 + \delta)$. In that case, $D^m g(x) \to 0^+$ as $x \to {x_0}^+$.

Under the hypotheses above, applying l'Hôpital's rule m times to the following *one-sided* limit eliminates the indeterminate form $0/0$ and yields

$$\lim_{x \to {x_0}^+} \frac{f(x)}{g(x)} = \lim_{x \to {x_0}^+} \frac{D^m f(x)}{D^m g(x)} = \frac{D^m f(x_0)}{\lim\limits_{x \to {x_0}^+} D^m g(x)} = \infty.$$

Using l'Hôpital's rule in this way proves that the one-sided limit of $f(x)/g(x)$ as $x \to {x_0}^+$ does not exist over the real numbers. We conclude that the original limit of $f(x)/g(x)$ as $x \to x_0$ also fails to exist. (Note that the original limit is one-sided if x_0 is the left endpoint of the interval X, and two-sided otherwise.) ■

Clearly, the multiplicity of zeros describes an important property of smooth functions. Building on this idea, the author proposes the following original terminology to describe two important properties of *sequences* of smooth functions:

Definition 8.4 *Let $X \subset \mathbb{R}$ be an interval, let $x_0 \in X$, and let n denote either a positive integer or infinity. Let $\{r_i\}_{i=0}^n \subset C^{n-1}(X)$ be an arbitrary sequence. If x_0 is a zero of r_i of multiplicity at least i for $0 \le i < n+1$, we say that the sequence $\{r_i\}_{i=0}^n$* ***is pre-l'Hôpital at*** *x_0. More explicitly, this means that*

$$D^i r_j(x_0) = 0 \quad \textit{for} \quad 0 \le i < j < n+1.$$

Note that this condition asserts nothing about the function r_0.

Now let $\{g_i\}_{i=0}^{n-1} \subset C^{n-1}(X)$ be an arbitrary sequence. If x_0 is a zero of g_i of multiplicity i for $0 \le i < n$, we say that the sequence $\{g_i\}_{i=0}^{n-1}$ ***is l'Hôpital at*** *x_0. More explicitly, this means that*

$$D^i g_j(x_0) = 0 \quad \text{for} \quad 0 \le i < j < n \qquad \text{and} \qquad D^i g_i(x_0) \ne 0 \quad \text{for} \quad 0 \le i < n.$$

In addition, if g_i is point-normalized at x_0 for $0 \le i < n$, we say that the sequence $\{g_i\}_{i=0}^{n-1}$ ***is point-normalized at*** *x_0, which means that*

$$D^i g_i(x_0) = 1 \quad \text{for} \quad 0 \le i < n.$$

The prominent role of l'Hôpital's rule in the proofs of this section amply justifies the terminology that we have introduced in his name. The reader may correctly infer from the choice of letters used above that we will be interested in *remainder* sequences $\{r_i\}_{i=0}^{n}$ that are pre-l'Hôpital at x_0 and in *asymptotic* sequences $\{g_i\}_{i=0}^{n-1}$ that are l'Hôpital at x_0. Clearly, every sequence which is l'Hôpital at x_0 is also pre-l'Hôpital at x_0. The following archetypal example illustrates the stronger of these two properties:

Example 8.5 *Let $X := \mathbb{R}$, let $x_0 \in \mathbb{R}$ be arbitrary, and let $n := \infty$. The sequence of polynomials $\{(x - x_0)^i\}_{i=0}^{\infty} \subset C^{\omega}(\mathbb{R})$ is l'Hôpital at x_0, and the scaled sequence $\{(x - x_0)^i/i!\}_{i=0}^{\infty}$ is also point-normalized at x_0. Note that $\{(x - x_0)^i\}_{i=0}^{\infty} \subset \mathrm{LNV}_{x_0}(\mathbb{R})$, and that $\{(x - x_0)^i\}_{i=0}^{\infty}$ is an asymptotic sequence as $x \to x_0$.*

The following proposition establishes that the properties of l'Hôpital sequences exhibited in the previous example also hold in general:

Proposition 8.6 *Let $X \subset \mathbb{R}$ be an interval, let $x_0 \in X$, and let n denote either a positive integer or infinity. If the sequence of functions $\{g_i\}_{i=0}^{n-1} \subset C^{n-1}(X)$ is l'Hôpital at x_0, then $\{g_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{x_0}(X)$, and $\{g_i(x)\}_{i=0}^{n-1}$ is an asymptotic sequence as $x \to x_0$.*

Proof. Since x_0 is a zero of g_i of multiplicity i, Proposition 8.2 implies that $g_i \in \mathrm{LNV}_{x_0}(X)$ for $0 \le i < n$. Now assume that $0 \le i < n-1$. Since x_0 is a zero of g_{i+1} of multiplicity $i+1$, it follows that x_0 is a zero of g_{i+1} of multiplicity at least i. Proposition 8.3 implies that the following limit exists and satisfies

$$\lim_{x \to x_0} \frac{g_{i+1}(x)}{g_i(x)} = \frac{D^i g_{i+1}(x_0)}{D^i g_i(x_0)} = 0.$$

Since this limit holds for $0 \leq i < n-1$, it follows that $\{g_i(x)\}_{i=0}^{n-1}$ is an asymptotic sequence as $x \to x_0$. ■

Let $f_1, f_2 \in C^n(X)$, where $n \in \mathbb{N}$ is arbitrary. We say that f_1 **Taylor interpolates** f_2 **to order** n **at** x_0 if

$$D^i f_1(x_0) = D^i f_2(x_0) \quad \text{for} \quad 0 \leq i \leq n.$$

This is clearly an equivalence relation on $C^n(X)$. Note that f_1 Taylor interpolates f_2 to order n at x_0 if and only if x_0 is a zero of $f_1 - f_2$ of multiplicity at least $n+1$; this equivalent reformulation is sometimes more convenient. The following archetypal example illustrates the Taylor interpolation scheme:

Example 8.7 *Given $f \in C^n(X)$ for arbitrary $n \in \mathbb{N}$, the Taylor polynomial of degree n for f at x_0 is*

$$T_n(x) := \sum_{i=0}^{n} \frac{D^i f(x_0)}{i!} \cdot (x - x_0)^i.$$

Since $D^i f(x_0) = D^i T_n(x_0)$ for $0 \leq i \leq n$, the polynomial T_n Taylor interpolates f to order n at x_0. Note also that the following asymptotic expansion holds:

$$f(x) \sim \sum_{i=0}^{n} \frac{D^i f(x_0)}{i!} \cdot (x - x_0)^i \quad \textit{as} \quad x \to x_0.$$

The previous example illustrates a principle which holds in general: We can perform Taylor interpolation at x_0 using an asymptotic expansion whose underlying asymptotic sequence is l'Hôpital at x_0. This technique motivates the following terminology: Let $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}$, where n is either a positive integer or infinity. If the sequence of functions $\{g_i\}_{i=0}^{n-1} \subset C^{n-1}(X)$ is l'Hôpital at x_0, we say that the asymptotic series

$$s_n := \sum_{i=0}^{n-1} c_i \cdot g_i$$

is l'Hôpital at x_0. Assume that $f \in C^{n-1}(X)$. If the asymptotic expansion

$$f(x) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x) \quad \text{as} \quad x \to x_0$$

holds under these hypotheses, we say that the asymptotic expansion **is l'Hôpital at** x_0. In addition, if the sequence $\{g_i\}_{i=0}^{n-1}$ is point-normalized at x_0, we say that the asymptotic series and asymptotic expansion **are point-normalized at** x_0.

If $n < \infty$, the asymptotic series s_n is a finite sum and satisfies $s_n \in C^{n-1}(X)$; in that case, it is clear what we mean if we say that s_n Taylor interpolates f to order $n-1$ at x_0. If $n = \infty$, the partial sums of the formal infinite series s_∞ satisfy $\{s_i\}_{i=0}^{\infty} \subset C^\infty(X)$; in that case, we say that s_∞ **Taylor interpolates** f **to order** ∞ **at** x_0 if s_i Taylor interpolates f to order $i-1$ at x_0 for all $i \geq 1$.

We have now established all of the definitions and fundamental results we will need to develop the theory of l'Hôpital asymptotic expansions in the next subsection.

8.1.2 The Theory of l'Hôpital Asymptotic Expansions

In this subsection, we will explore the connections between the many univariate constructs defined in the previous subsection. The following fundamental theorem explains how to perform Taylor interpolation at x_0 using asymptotic series which are both l'Hôpital and point-normalized at x_0:

Theorem 8.8 (Univariate Taylor Interpolation) *Let $X \subset \mathbb{R}$ be an interval, let $x_0 \in X$, and let $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}$, where n is either a positive integer or infinity. Assume that $f \in C^{n-1}(X)$ and that the sequence $\{g_i\}_{i=0}^{n-1} \subset C^{n-1}(X)$ is both l'Hôpital and point-normalized at x_0. The following four statements are equivalent:*

1. *The point-normalized l'Hôpital asymptotic series*

$$s_n := \sum_{i=0}^{n-1} c_i \cdot g_i$$

Taylor interpolates f to order $n-1$ at x_0.

2. *The coefficients $\{c_i\}_{i=0}^{n-1}$ can be calculated recursively from the remainders $\{r_i\}_{i=0}^{n}$ via*

$$c_i = D^i r_i(x_0) \quad \text{for} \quad 0 \leq i < n. \tag{8.1}$$

3. *The sequence of remainders $\{r_i\}_{i=0}^{n}$ is pre-l'Hôpital at x_0.*

4. *The following point-normalized l'Hôpital asymptotic expansion holds:*

$$f(x) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x) \quad as \quad x \to x_0.$$

Proof. By definition of the case $n = \infty$, it suffices to show that the theorem is true for all positive integers n. For the rest of the proof, let n denote an arbitrary positive integer.

Recall that since $\{g_i\}_{i=0}^{n-1} \subset C^{n-1}(X)$ is both l'Hôpital and point-normalized at x_0 by assumption, it follows that $D^i g_j(x_0) = 0$ for $0 \le i < j \le n-1$, that $D^i g_i(x_0) = 1$ for $0 \le i \le n-1$, that $\{g_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{x_0}(X)$, and that $\{g_i(x)\}_{i=0}^{n-1}$ is an asymptotic sequence as $x \to x_0$. Note also that $\{s_i\}_{i=0}^{n} \cup \{r_i\}_{i=0}^{n} \subset C^{n-1}(X)$. Finally, note that $0 \le i < j \le n$ implies

$$\begin{aligned} D^i s_j(x_0) &= \sum_{k=0}^{j-1} c_k \cdot D^i g_k(x_0) = \sum_{k=0}^{i} c_k \cdot D^i g_k(x_0) = D^i s_i(x_0) + c_i, \\ D^i r_j(x_0) &= D^i f(x_0) - D^i s_j(x_0) = D^i f(x_0) - (D^i s_i(x_0) + c_i) = D^i r_i(x_0) - c_i. \end{aligned}$$

We will use these facts freely throughout the proof.

By definition, the asymptotic series s_n Taylor interpolates f to order $n-1$ at x_0 if and only if

$$D^i f(x_0) = D^i s_n(x_0) = D^i s_i(x_0) + c_i \quad \text{for} \quad 0 \le i \le n-1.$$

This is a triangular system of n linear equations in the n unknowns $\{c_i\}_{i=0}^{n-1}$, and has a unique solution given recursively by

$$c_i = D^i f(x_0) - D^i s_i(x_0) = D^i r_i(x_0) \quad \text{for} \quad 0 \le i \le n-1.$$

The above solution holds if and only if

$$D^i r_j(x_0) = D^i r_i(x_0) - c_i = 0 \quad \text{for} \quad 0 \le i < j \le n,$$

which asserts that $\{r_i\}_{i=0}^{n}$ is pre-l'Hôpital at x_0. Since the sequence $\{g_i\}_{i=0}^{n-1}$ is l'Hôpital at x_0, Proposition 8.3 implies that the sequence $\{r_i\}_{i=0}^{n}$ is pre-l'Hôpital at x_0 if and only if the following limits exist and satisfy

$$\lim_{x \to x_0} \frac{r_{i+1}(x)}{g_i(x)} = \frac{D^i r_{i+1}(x_0)}{D^i g_i(x_0)} = \frac{0}{1} = 0 \quad \text{for} \quad 0 \le i \le n-1.$$

By the definition of an asymptotic expansion, these limits exist if and only if the following asymptotic expansion holds:

$$f(x) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x) \quad \text{as} \quad x \to x_0.$$

This proves that all four statements of the theorem are equivalent. ■

The previous theorem states that an n-term asymptotic series which is both l'Hôpital and point-normalized at x_0 will Taylor interpolate a function f to order $n-1$ at x_0 if and only if the l'Hôpital asymptotic series is a l'Hôpital asymptotic expansion of f to n terms at x_0. The theorem also provides an equivalent condition that tells us how to calculate the coefficients $\{c_i\}_{i=0}^{n-1}$ of the l'Hôpital asymptotic expansion by evaluating the higher-order derivatives of the remainders $\{r_i\}_{i=0}^{n-1}$ at x_0. The following example uses the method of the theorem to calculate the coefficients of the Taylor polynomial:

Example 8.9 *We now reformulate Example 8.7 as follows: Let n denote a positive integer, and assume that $f \in C^{n-1}(X)$. The Taylor polynomial of degree $n-1$ for f at x_0 is*

$$T_{n-1}(x) := \sum_{i=0}^{n-1} D^i f(x_0) \cdot \frac{(x-x_0)^i}{i!}.$$

The polynomial T_{n-1} Taylor interpolates f to order $n-1$ at x_0. In fact, $s_n := T_{n-1}$ is the asymptotic expansion of f to n terms at x_0 with respect to the asymptotic sequence $\{(x-x_0)^i/i!\}_{i=0}^{n-1}$, which is both l'Hôpital and point-normalized at x_0. By Univariate Taylor Interpolation Theorem 8.8, we can calculate the coefficients $\{c_i\}_{i=0}^{n-1}$ recursively from the remainders $\{r_i\}_{i=0}^{n}$ via

$$c_i = D^i r_i(x_0) = D^i f(x_0) - D^i T_{i-1}(x_0) = D^i f(x_0) \quad \textit{for} \quad 0 \le i \le n-1.$$

This result agrees with the coefficients of the Taylor polynomial T_{n-1} given above.

The successful use of Taylor polynomials, Padé approximations, and similar interpolation schemes in applications is based on the fact that univariate Taylor interpolation generally yields a good local approximation of the original function in a neighborhood of the specified point. In the next subsection, we will learn how to estimate this approximation error rigorously and will describe its asymptotic behavior.

8.1.3 Remainder and Convergence Theories

In this subsection, we will develop the remainder theory for Taylor interpolation by a *general* smooth function of one real variable, and we will use this remainder theory to give a sufficient condition for the uniform convergence of l'Hôpital asymptotic expansions with infinitely many terms. We begin by recalling Taylor's theorem from elementary calculus, which describes the remainder for Taylor interpolation by a very *special* smooth function—*the Taylor polynomial.*

Let $X \subset \mathbb{R}$ be an interval, let $x_0 \in X$, and assume that $f \in C^n(X)$, where n is a positive integer. Let T_{n-1} denote the Taylor polynomial of degree $n-1$ for f at x_0. Taylor's theorem in integral form asserts that the remainder satisfies

$$f(x) - T_{n-1}(x) = \frac{1}{(n-1)!} \cdot \int_{x_0}^{x} (x-s)^{n-1}\, D^n f(s)\, ds \quad \text{for all} \quad x \in X.$$

Taylor's theorem in mean-value form asserts that for each $x_1 \in X \backslash \{x_0\}$, there exists some $\xi \in X$ such that

$$f(x_1) - T_{n-1}(x_1) = \frac{D^n f(\xi)}{n!} \cdot (x_1 - x_0)^n \quad \text{and} \quad \min x_i < \xi < \max x_i.$$

If $\|D^n f\|_\infty < \infty$, then the mean-value form of the remainder yields the rigorous error estimate

$$|f(x) - T_{n-1}(x)| \leq \frac{\|D^n f\|_\infty}{n!} \cdot |x - x_0|^n \quad \text{for all} \quad x \in X,$$

which in turn implies the following asymptotic behavior:

$$f(x) - T_{n-1}(x) = O\left((x - x_0)^n\right) \quad \text{for all} \quad x \in X.$$

In addition, if $|x - x_0| \leq \rho$ for all $x \in X$, then we obtain the following uniform error estimate:

$$\|f - T_{n-1}\|_\infty \leq \frac{\|D^n f\|_\infty}{n!} \cdot \rho^n.$$

These standard elementary results yield the following general remainder theorem for Taylor interpolation:

Theorem 8.10 (Univariate Taylor Remainder) *Let $X \subset \mathbb{R}$ be an interval, let $x_0 \in X$, and assume that $f_1, f_2 \in C^n(X)$, where n is a positive integer. If f_1 Taylor interpolates*

f_2 to order $n-1$ at x_0, then the remainder $r := f_1 - f_2$ satisfies

$$r(x) = \frac{1}{(n-1)!} \cdot \int_{x_0}^{x} (x-s)^{n-1} \, D^n r(s) \, ds \quad \textit{for all} \quad x \in X.$$

In addition, for each $x_1 \in X \backslash \{x_0\}$, there exists some $\xi \in X$ such that

$$r(x_1) = \frac{D^n r(\xi)}{n!} \cdot (x_1 - x_0)^n \quad \textit{and} \quad \min x_i < \xi < \max x_i.$$

If $\|D^n r\|_\infty < \infty$, then we obtain the rigorous error estimate

$$|r(x)| \leq \frac{\|D^n r\|_\infty}{n!} \cdot |x - x_0|^n \quad \textit{for all} \quad x \in X,$$

which in turn implies the following asymptotic behavior:

$$r(x) = O\left((x - x_0)^n\right) \quad \textit{for all} \quad x \in X.$$

In addition, if $|x - x_0| \leq \rho$ for all $x \in X$, then we obtain the following uniform error estimate:

$$\|r\|_\infty \leq \frac{\|D^n r\|_\infty}{n!} \cdot \rho^n.$$

Proof. Since f_1 Taylor interpolates f_2 to order $n-1$ at x_0, it follows that x_0 is a zero of the remainder r of multiplicity at least n. This means that the Taylor polynomial of degree $n-1$ for r at x_0 is identically zero. The conclusions of the theorem follow by applying Taylor's theorem to the function r with the Taylor polynomial $T_{n-1} = 0$. ∎

Since the n-th derivative of a polynomial of degree $n-1$ is identically zero, the previous theorem actually includes Taylor's theorem as a special case. The previous theorem is more general than Taylor's theorem because it accommodates Taylor interpolation by functions which are not necessarily polynomials.

We will now use the previous theorem to develop the following sufficient condition for uniform convergence on compact intervals for l'Hôpital asymptotic expansions with infinitely many terms:

Theorem 8.11 (Univariate Taylor Convergence) *Let $[a, b] \subset \mathbb{R}$ be a compact interval, and let $x_0 \in [a, b]$. Assume that $f \in C^\infty[a, b]$ has the following l'Hôpital asymptotic expansion*

$$f(x) \sim \sum_{i=0}^{\infty} c_i \cdot g_i(x) \quad \textit{as} \quad x \to x_0,$$

and let $\{r_n\}_{n=0}^{\infty} \subset C^{\infty}[a,b]$ *denote the sequence of remainders. If there exists a real constant* $\gamma > 0$ *such that*

$$\|D^n r_n\|_{\infty} = O(\gamma^n) \quad \text{as} \quad n \to \infty,$$

then the formal infinite series

$$s_{\infty} := \sum_{i=0}^{\infty} c_i \cdot g_i$$

converges uniformly on $[a,b]$*, and the sum of the infinite series* s_{∞} *is the function* f*.*

Proof. We can assume without loss of generality that the given l'Hôpital asymptotic expansion is point-normalized at x_0. Univariate Taylor Interpolation Theorem 8.8 implies that the point-normalized l'Hôpital asymptotic expansion s_{∞} Taylor interpolates f to order ∞ at x_0, which by definition means that for all $n \geq 1$, the partial sum s_n Taylor interpolates f to order $n-1$ at x_0. Since $r_n \in C^{\infty}[a,b]$ and $[a,b]$ is compact, we know that $\|D^n r_n\|_{\infty} < \infty$. Univariate Taylor Remainder Theorem 8.10 with $\rho := b-a$ therefore yields the uniform error estimate

$$\|r_n\|_{\infty} \leq \frac{\|D^n r_n\|_{\infty}}{n!} \cdot (b-a)^n \quad \text{for all} \quad n \geq 0.$$

Since $\|D^n r_n\|_{\infty} = O(\gamma^n)$ as $n \to \infty$, there exists a real constant $M > 0$ and a natural number N such that

$$\|D^n r_n\|_{\infty} \leq M \cdot \gamma^n \quad \text{for all} \quad n \geq N.$$

The previous two inequalities together imply that

$$\|r_n\|_{\infty} \leq M \cdot \frac{[\gamma \cdot (b-a)]^n}{n!} \quad \text{for all} \quad n \geq N.$$

This new inequality and the limit

$$\lim_{n \to \infty} \frac{[\gamma \cdot (b-a)]^n}{n!} = 0$$

together imply that $\|r_n\|_{\infty} \to 0$ as $n \to \infty$. We conclude that $s_n \to f$ uniformly on $[a,b]$ as $n \to \infty$. ■

The previous theorem tells us that even if the sequence $\{\|D^n r_n\|_{\infty}\}_{n=0}^{\infty}$ grows at a exponential rate, the formal infinite series s_{∞} is guaranteed to converge uniformly to f on $[a,b]$. Of course, if the sequence $\{\|D^n r_n\|_{\infty}\}_{n=0}^{\infty}$ is bounded, or grows at a logarithmic or polynomial rate, we can also conclude that the infinite series s_{∞} converges uniformly to f

on $[a, b]$.

This completes our development of the theory of Taylor interpolation for functions of one real variable. Later in this chapter, we will explore the implications of the univariate theory as we develop the bivariate theory of Taylor interpolation using dual asymptotic expansions as our primary tool.

8.2 A Bridge between One and Two Dimensions

The results in this section form a bridge between the univariate and bivariate theories of Taylor interpolation. In the first subsection, we will abstract the special structure of the kinds of logical propositions which arise frequently in the bivariate theory of Taylor interpolation. This abstract logical framework will allow us to develop a new proof technique which we will use later as a labor-saving device. In the second subsection, we will develop a little-known but very useful version of Rolle's theorem for smooth functions on bounded rectangles. We will use this geometrically-motivated theorem later in the chapter to deduce the far-reaching implications of zero lines for smooth functions of two real variables.

8.2.1 Logical Parametric Extensions and Logical Tensor Products

Let X and Y be arbitrary sets, let $x_0 \in X$ and $y_0 \in Y$, and assume that $f : X \times Y \to F$, where F is an arbitrary field. In Section 5.1, we called $x = x_0$ a **zero line of** f if $f(x_0, y) = 0$ for all $y \in Y$. Similarly, we called $y = y_0$ a **zero line of** f if $f(x, y_0) = 0$ for all $x \in X$.

The *zero lines* of a function of two variables are actually the *parametric extensions of the ordinary zeros* of a function of one variable in the following sense: If x_0 is a zero of some univariate function of x, it means that this univariate function is annihilated by the evaluation functional ε_{x_0}. If $x = x_0$ is a zero line of some bivariate function of x and y, it means that this bivariate function is annihilated by the evaluation operator E_{x_0}, which is by definition the parametric extension of ε_{x_0}. A similar relationship holds between the zero y_0, the evaluation functional ε^{y_0}, the zero line $y = y_0$, and the evaluation operator E^{y_0}.

We will find it very useful to develop the notion of parametric extensions for *propositions about functions* into a formal logical framework. This will lead to a new formal proof technique that allows us to automatically deduce theorems about a special class of bivariate propositions from related theorems about univariate propositions. The above example concerning zero lines motivates the following original terminology and notation:

Definition 8.12 *Let X, Y, and Z be arbitrary sets, and let x_0 and y_0 be arbitrary symbols. Assume that $u : X \to Z$, that $v : Y \to Z$, and that $f : X \times Y \to Z$. Let $P(x_0; u)$ denote a logical proposition which asserts some relationship between the symbol x_0 and the univariate function u. We define the* ***logical parametric extension of the proposition*** *$P(x_0; u)$ to be the proposition which asserts that $P(x_0; f^y)$ holds for all $y \in Y$, and we denote this proposition by $\tilde{P}(x_0; f)$. Similarly, let $Q(y_0; v)$ denote a logical proposition which asserts some relationship between the symbol y_0 and the univariate function v. We define the* ***logical parametric extension of the proposition*** *$Q(y_0; v)$ to be the proposition which asserts that $Q(y_0; f_x)$ holds for all $x \in X$, and we denote this proposition by $\tilde{Q}(y_0; f)$. We define the* ***logical tensor product of the propositions*** *$\tilde{P}(x_0; f)$* ***and*** *$\tilde{Q}(y_0; f)$ to be the proposition which asserts that both $\tilde{P}(x_0; f)$ and $\tilde{Q}(y_0; f)$ hold simultaneously, and we denote this proposition by $(\tilde{P} \otimes \tilde{Q})(x_0, y_0; f)$.*

We can express the above definitions more concisely in logical notation as follows:

$$\begin{aligned} \tilde{P}(x_0; f) \quad &\Leftrightarrow \quad P(x_0; f^y) \ \forall y \in Y \\ \tilde{Q}(y_0; f) \quad &\Leftrightarrow \quad Q(y_0; f_x) \ \forall x \in X \\ (\tilde{P} \otimes \tilde{Q})(x_0, y_0; f) \quad &\Leftrightarrow \quad \tilde{P}(x_0; f) \ \&\ \tilde{Q}(y_0; f). \end{aligned}$$

Note that the logical tensor product allows us to construct a *bivariate proposition* in the symbols x_0 and y_0 which asserts some property of a *bivariate function* f. The name "logical tensor product" seems appropriate since the logical operation "&" is the logical equivalent of multiplication; in fact, if we model Boolean logic by the Galois field with two elements, logical "&" actually *is* multiplication in the field.

How do these formal definitions capture the sense in which the zero lines of bivariate functions are parametric extensions of the ordinary zeros of univariate functions? Let $P(x_0; u)$ denote the proposition that x_0 is a zero of the univariate function u. Under these definitions, the logical parametric extension $\tilde{P}(x_0; f)$ is the proposition that $x = x_0$ is a zero line of the bivariate function f. Similarly, if $Q(y_0; v)$ denotes the proposition that y_0 is a zero of the univariate function v, the logical parametric extension $\tilde{Q}(y_0; f)$ is the proposition that $y = y_0$ is a zero line of the bivariate function f. From the logical parametric extensions $\tilde{P}(x_0; f)$ and $\tilde{Q}(y_0; f)$, which are both *univariate* propositions, we can construct the logical tensor product $(\tilde{P} \otimes \tilde{Q})(x_0, y_0; f)$, which is the *bivariate* proposition which asserts that both $x = x_0$ and $y = y_0$ are zero lines of the bivariate function f.

In actual practice, the univariate propositions $P(x_0; u)$ and $Q(y_0; v)$ may have more parameters than merely the univariate functions u and v. The best example of this comes from the definition of a dual asymptotic expansion:

Example 8.13 *Let $P(x_0;\ u,\ \{c_i\}_{i=0}^{n-1},\ \{g_i\}_{i=0}^{n-1})$ denote the proposition that the following univariate asymptotic expansion holds:*

$$u(x) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x) \quad \textit{as} \quad x \to x_0.$$

Similarly, let $Q(y_0;\ v,\ \{c_i\}_{i=0}^{n-1},\ \{h_i\}_{i=0}^{n-1})$ denote the proposition that the following univariate asymptotic expansion holds:

$$v(y) \sim \sum_{i=0}^{n-1} c_i \cdot h_i(y) \quad \textit{as} \quad y \to y_0.$$

If we replace the coefficients by coefficient functions*, we can construct the logical parametric extensions*

$$\tilde{P}(x_0;\ f,\ \{c_i \cdot h_i\}_{i=0}^{n-1},\ \{g_i\}_{i=0}^{n-1}) \quad \textit{and} \quad \tilde{Q}(y_0;\ f,\ \{c_i \cdot g_i\}_{i=0}^{n-1},\ \{h_i\}_{i=0}^{n-1}),$$

which we interpret to mean that the following two univariate asymptotic expansions hold for all values of the parameters:

$$\begin{aligned} f^y(x) &\sim \sum_{i=0}^{n-1} (c_i \cdot h_i(y)) \cdot g_i(x) \quad \textit{as} \quad x \to x_0 \quad \textit{for all} \quad y \in Y \\ f_x(y) &\sim \sum_{i=0}^{n-1} (c_i \cdot g_i(x)) \cdot h_i(y) \quad \textit{as} \quad y \to y_0 \quad \textit{for all} \quad x \in X. \end{aligned}$$

If we assert that both of these logical parametric extensions hold simultaneously, we obtain the logical tensor product

$$(\tilde{P} \otimes \tilde{Q})(x_0, y_0;\ f,\ \{c_i\}_{i=0}^{n-1},\ \{g_i\}_{i=0}^{n-1},\ \{h_i\}_{i=0}^{n-1}),$$

which by Definition 7.20 asserts that the following dual asymptotic expansion holds:

$$f(x,y) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x)\, h_i(y) \quad as \quad x \to x_0 \quad or \quad y \to y_0.$$

We conclude that the bivariate proposition which defines a dual asymptotic expansion has a very special structure: It is the logical tensor product of the logical parametric extensions of two univariate propositions which define univariate asymptotic expansions modulo suitable parameters.

The following remark explains the choice of parameters for the logical tensor product $\tilde{P} \otimes \tilde{Q}$ constructed in the previous example:

Remark 8.14 *We adopt the convention that the logical tensor product $\tilde{P} \otimes \tilde{Q}$ uses the simplest collection of parameters which allows us to reconstruct the original parameters of the two factors $\tilde{P}$ and $\tilde{Q}$.*

In order to articulate a new formal proof technique for bivariate propositions with the special structure illustrated in the previous example, we will need the following original terminology:

Definition 8.15 *Let X, Y, and Z be arbitrary sets, and let $U \subset Z^X$ and $V \subset Z^Y$ and $W \subset Z^{X \times Y}$ be arbitrary subsets. We say that the triple (U, V, W)* ***is cross-sectionally compatible*** *if $f^y \in U$ and $f_x \in V$ for all $f \in W$, $x \in X$, and $y \in Y$.*

For example, the triple $(Z^X,\ Z^Y,\ Z^{X \times Y})$ is always cross-sectionally compatible. If X and Y are topological spaces, the triple $(C(X),\ C(Y),\ C(X \times Y))$ is cross-sectionally compatible as well. The following theorem describes a new formal proof technique for the logical tensor product of logical parametric extensions, assuming cross-sectional compatibility:

Theorem 8.16 (Logical Tensor Product Implication) *Let X, Y, and Z be arbitrary sets, and let x_0 and y_0 be arbitrary symbols. Let $U \subset Z^X$ and $V \subset Z^Y$ and $W \subset Z^{X \times Y}$ be arbitrary subsets, and assume that the triple (U, V, W) is cross-sectionally compatible. For $i = 1, 2$, let $P_i(x_0; u)$ and $Q_i(y_0; v)$ denote logical propositions which are well-defined for all $u \in U$ and $v \in V$. If the implication*

$$P_1(x_0; u) \quad \Rightarrow \quad P_2(x_0; u) \quad holds \quad \forall u \in U,$$

and the implication

$$Q_1(y_0; v) \quad \Rightarrow \quad Q_2(y_0; v) \quad \textit{holds} \quad \forall v \in V,$$

then the implication

$$(\tilde{P}_1 \otimes \tilde{Q}_1)(x_0, y_0; f) \quad \Rightarrow \quad (\tilde{P}_2 \otimes \tilde{Q}_2)(x_0, y_0; f) \quad \textit{holds} \quad \forall f \in W.$$

Proof. Let $f \in W$ be arbitrary. We can express the entire proof concisely in logical notation as follows:

$$\begin{aligned}
(\tilde{P}_1 \otimes \tilde{Q}_1)(x_0, y_0; f) \quad &\Leftrightarrow \quad \tilde{P}_1(x_0; f) \;\&\; \tilde{Q}_1(y_0; f) \\
&\Leftrightarrow \quad [P_1(x_0; f^y)\ \forall y \in Y] \;\&\; [Q_1(y_0; f_x)\ \forall x \in X] \\
&\Rightarrow \quad [P_2(x_0; f^y)\ \forall y \in Y] \;\&\; [Q_2(y_0; f_x)\ \forall x \in X] \\
&\Leftrightarrow \quad \tilde{P}_2(x_0; f) \;\&\; \tilde{Q}_2(y_0; f) \\
&\Leftrightarrow \quad (\tilde{P}_2 \otimes \tilde{Q}_2)(x_0, y_0; f).
\end{aligned}$$

This completes the proof. ■

In the case of bidirectional implications, the previous theorem immediately yields the following corollary:

Corollary 8.17 (Logical Tensor Product Equivalence) *Assume the hypotheses of Logical Tensor Product Implication Theorem 8.16. If the equivalence*

$$P_1(x_0; u) \quad \Leftrightarrow \quad P_2(x_0; u) \quad \textit{holds} \quad \forall u \in U,$$

and the equivalence

$$Q_1(y_0; v) \quad \Leftrightarrow \quad Q_2(y_0; v) \quad \textit{holds} \quad \forall v \in V,$$

then the equivalence

$$(\tilde{P}_1 \otimes \tilde{Q}_1)(x_0, y_0; f) \quad \Leftrightarrow \quad (\tilde{P}_2 \otimes \tilde{Q}_2)(x_0, y_0; f) \quad \textit{holds} \quad \forall f \in W.$$

The formal logical framework we have developed in this subsection will reassert itself many times in the logical structure of many definitions which will appear in later in this chapter. This in turn will enable us to use Logical Tensor Product Equivalence Corollary

8.17 to develop a bivariate analogue of Univariate Taylor Interpolation Theorem 8.8 with only a small amount of additional work!

8.2.2 Rolle's Theorem on Bounded Rectangles

We now leave the subject of mathematical logic and turn our attention to differential calculus in two real variables. We will begin by defining some very useful geometric terminology.

Recall that in Section 5.1, we called the set $X \times Y$ a **rectangle**, where X and Y were *arbitrary* sets. We will now assume that $X, Y \subset \overline{\mathbb{R}}$ are *intervals*. Let $x_0, x_1 \in \overline{\mathbb{R}}$ be the endpoints of the interval X, and let $y_0, y_1 \in \overline{\mathbb{R}}$ be the endpoints of the interval Y. We call the four points (x_i, y_j) for $0 \leq i, j \leq 1$ the **vertices of the rectangle** $X \times Y$. Conversely, given distinct $x_0, x_1 \in \overline{\mathbb{R}}$ and distinct $y_0, y_1 \in \overline{\mathbb{R}}$, let $\underline{x} := \min x_i$ and $\overline{x} := \max x_i$, and let $\underline{y} := \min y_i$ and $\overline{y} := \max y_i$. We call the set $R := (\underline{x}, \overline{x}) \times (\underline{y}, \overline{y})$ the **open rectangle with vertices** $\{(x_i, y_j) : 0 \leq i, j \leq 1\}$, and we call its closure $\overline{R} = [\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}]$ the **closed rectangle with vertices** $\{(x_i, y_j) : 0 \leq i, j \leq 1\}$. Note that the *open* rectangle R does *not* contain its vertices, but the *closed* rectangle $\overline{R}$ does.

In order to avoid imposing unnecessarily strong hypotheses, we need to introduce some new terminology and notation to describe the order of differentiability of real-valued functions of two real variables:

Definition 8.18 *Let $X, Y \subset \mathbb{R}$ be intervals, let $\ell, m \in \mathbb{N}$, and assume that $f : X \times Y \to \mathbb{R}$. If for all i and j satisfying $0 \leq i \leq \ell$ and $0 \leq j \leq m$, the mixed $(i+j)$-th order partial derivatives $\partial_{z_1}\partial_{z_2}\cdots\partial_{z_{i+j}} f$ all exist and are continuous on $X \times Y$ for all permutations $z_1, z_2, \ldots, z_{i+j}$ of i copies of x and j copies of y, we say that f* ***is continuously differentiable to order*** *(ℓ, m). We will denote the set of all real-valued functions on $X \times Y$ which are continuously differentiable to order (ℓ, m) by $C^{(\ell,m)}(X \times Y)$. In particular, we define $C^{(0,0)}(X \times Y) := C(X \times Y)$.*

The set $C^{(\ell,m)}(X \times Y)$ is a function space over $\mathbb{R}$. Note that the subspace inclusion $C^{\ell+m}(X \times Y) \subset C^{(\ell,m)}(X \times Y)$ always holds by definition; however, the reverse inclusion need not hold. For example, the function defined by $f(x, y) := x\,|x| \cdot y\,|y|$ for all $(x, y) \in \mathbb{R}^2$ satisfies $f \in C^{(1,1)}\left(\mathbb{R}^2\right)$, but $f \notin C^2\left(\mathbb{R}^2\right)$.

Remark 8.19 *The weaker hypothesis $f \in C^{(\ell,m)}(X \times Y)$ allows us to specify the order of differentiability in each variable separately, whereas the stronger hypothesis $f \in C^{\ell+m}(X \times Y)$ specifies the total order of differentiability for the two variables together.*

For example, $f \in C^{(1,1)}(X \times Y)$ means that f and $\partial_x f$ and $\partial_y f$ and $\partial_x \partial_y f$ and $\partial_y \partial_x f$ all exist and are continuous on $X \times Y$; according to [Rud76, pp. 235–236], these hypotheses are more than sufficient to imply that $\partial_x \partial_y f = \partial_y \partial_x f$ on $X \times Y$. Furthermore, given $f \in C^{(\ell,m)}(X \times Y)$, and given any i and j satisfying $0 \le i \le \ell$ and $0 \le j \le m$, the mixed $(i+j)$-th order partial derivatives $\partial_{z_1} \partial_{z_2} \cdots \partial_{z_{i+j}} f$ are *identical* for all permutations z_1, z_2, ..., z_{i+j} of i copies of x and j copies of y; since every permutation is the product of transpositions, this follows by repeated applications of the result for mixed second-order partial derivatives. We conclude that the traditional hypothesis $f \in C^{\ell+m}(X \times Y)$ is stronger than necessary to obtain the commuting of mixed partial derivatives up to order ℓ in x and up to order m in y; the hypothesis $f \in C^{(\ell,m)}(X \times Y)$ will suffice.

In general, the weaker hypothesis $f \in C^{(\ell,m)}(X \times Y)$ is better suited for the methods of this thesis than the stronger hypothesis $f \in C^{\ell+m}(X \times Y)$. For example, if $f \in C^{(\ell,m)}(X \times Y)$, the cross-sections satisfy $f^y \in C^\ell(X)$ and $f_x \in C^m(Y)$ for all $x \in X$ and $y \in Y$, which means that the triple $(C^\ell(X),\ C^m(Y),\ C^{(\ell,m)}(X \times Y))$ is cross-sectionally compatible. Conversely, given two univariate functions $g \in C^\ell(X)$ and $h \in C^m(Y)$, their tensor product satisfies $g \otimes h \in C^{(\ell,m)}(X \times Y)$.

Using the new terminology and notation of this subsection, we can now state and prove a version of Rolle's theorem for smooth functions on bounded rectangles. This somewhat surprising but rather delightful result will help us to deduce some of the important implications of zero lines for smooth functions of class $C^{(1,1)}$, and more generally, of class $C^{(n,n)}$:

Proposition 8.20 *Given distinct $x_0, x_1 \in \mathbb{R}$ and distinct $y_0, y_1 \in \mathbb{R}$, let R denote the open rectangle with vertices $\{(x_i, y_j) : 0 \le i, j \le 1\}$. Assume that $f \in C^{(1,1)}(\bar{R})$. If f vanishes at all four vertices of R, then there exists a point $(\xi, \eta) \in R$ such that $(\partial_x \partial_y f)(\xi, \eta) = 0$.*

Proof. By assumption, $f(x_i, y_j) = 0$ for $0 \le i, j \le 1$. Let $X := (\min x_i, \max x_i)$ and $Y := (\min y_i, \max y_i)$ denote open intervals, and note that $R := X \times Y$ and $\bar{R} = \bar{X} \times \bar{Y}$. The key to this proof is to define a pair of univariate functions on the closed intervals $\bar{X}$ and $\bar{Y}$ using a clever technique found in [Rud76, p. 235]. We define the first univariate function $u \in C^1(\bar{X})$ by $u(x) := f(x, y_1) - f(x, y_0)$ for all $x \in \bar{X}$. Since $u(x_0) = u(x_1) = 0$, Rolle's theorem implies that there exists $\xi \in X$ such that

$$Du(\xi) = \partial_x f(\xi, y_1) - \partial_x f(\xi, y_0) = 0.$$

We now use the value of ξ obtained from u to define the second univariate function $v \in$

$C^1(\bar{Y})$ by $v(y) := \partial_x f(\xi, y) - \partial_x f(\xi, y_0)$ for all $y \in \bar{Y}$. Since $v(y_0) = v(y_1) = 0$, Rolle's theorem implies that there exists $\eta \in Y$ such that

$$Dv(\eta) = (\partial_x \partial_y f)(\xi, \eta) = 0.$$

We conclude that $(\xi, \eta) \in X \times Y =: R$, as desired. ■

How does Rolle's theorem for rectangles help us understand the implications of zero lines? The following corollary describes one such implication:

Corollary 8.21 *Let $X, Y \subset \mathbb{R}$ be intervals, and let $x_0 \in X$ and $y_0 \in Y$. Assume that $f \in C^{(1,1)}(X \times Y)$. If $x = x_0$ and $y = y_0$ are zero lines of f, and if $(x_1, y_1) \in X \times Y$ is a zero of f which does not lie on either of these zero lines, then $\partial_x \partial_y f$ must vanish at some point (ξ, η) which lies in the open rectangle with vertices $\{(x_i, y_j) : 0 \leq i, j \leq 1\}$.*

The next section will explore further implications of the previous corollary in the context of the Taylor interpolation of functions of two real variables.

8.3 Taylor Interpolation on Two Lines in the Plane

In this section, we will develop bivariate analogues of the foundational notions of multiple zeros and l'Hôpital sequences of functions. We will then parametrically extend the univariate theory of Taylor interpolation at a point on the real line in order to systematically develop a bivariate theory of Taylor interpolation on two rectilinear coordinate lines in the plane.

8.3.1 Definitions and Fundamental Results

We now extend the definition of zero lines to include multiplicity. Let $X, Y \subset \mathbb{R}$ be intervals, and let $x_0 \in X$ and $y_0 \in Y$. Assume that $f \in C^{(n,n)}(X \times Y)$, where $n \in \mathbb{N}$ is arbitrary. We say that **$x = x_0$ and $y = y_0$ are zero lines of f of multiplicity at least n** if the higher-order normal derivatives of f on the lines $x = x_0$ and $y = y_0$ satisfy

$$E_{x_0} \partial_x^i f = 0 \quad \text{and} \quad E^{y_0} \partial_y^i f = 0 \quad \text{for} \quad 0 \leq i < n.$$

This property is vacuously true whenever $n = 0$. We say that **$x = x_0$ and $y = y_0$ are zero lines of f of multiplicity (exactly) n** if $x = x_0$ and $y = y_0$ are zero lines of f of multiplicity at least n and

$$(\partial_x^n \partial_y^n f)(x_0, y_0) \neq 0.$$

In particular, when we assert that $x = x_0$ and $y = y_0$ are zero lines of f of multiplicity 0, we are saying that $f(x_0, y_0) \neq 0$, which means that $x = x_0$ and $y = y_0$ are *not* zero lines of f.

Remark 8.22 *The bivariate proposition which asserts that $x = x_0$ and $y = y_0$ are zero lines of $f \in C^{(n,n)}(X \times Y)$ of multiplicity at least n has a special logical structure: It is the logical tensor product of the logical parametric extensions of two univariate propositions— the proposition which asserts that x_0 is a zero of $u \in C^n(X)$ of multiplicity at least n and the proposition which asserts that y_0 is a zero of $v \in C^n(Y)$ of multiplicity at least n.*

Zero lines with multiplicity yield a useful *univariate identity property* which has no analogue in the case of ordinary zeros with multiplicity. By definition, if $x = x_0$ and $y = y_0$ are zero lines of f of multiplicity at least n, then the following *univariate identities* hold:

$$\partial_x^i f(x_0, y) \equiv 0 \quad \text{and} \quad \partial_y^i f(x, y_0) \equiv 0 \quad \text{for} \quad 0 \leq i < n.$$

If $n \geq 1$, this collection of identities is nonempty, and we can *differentiate* these univariate identities j times each to obtain the following *enlarged* collection of identities:

$$(\partial_y^j \partial_x^i f)(x_0, y) \equiv 0 \quad \text{and} \quad (\partial_x^j \partial_y^i f)(x, y_0) \equiv 0 \quad \text{for} \quad 0 \leq i < n \quad \text{and} \quad 0 \leq j \leq n.$$

In particular, if $j = 1$, we obtain the following (possibly empty) subcollection of univariate identities:

$$\partial_x^{i-1}(\partial_x \partial_y f)(x_0, y) \equiv 0 \quad \text{and} \quad \partial_y^{i-1}(\partial_x \partial_y f)(x, y_0) \equiv 0 \quad \text{for} \quad 1 \leq i < n.$$

This subcollection asserts that $x = x_0$ and $y = y_0$ are zero lines of $\partial_x \partial_y f \in C^{(n-1,n-1)}(X \times Y)$ of multiplicity at least $n-1$. (This holds vacuously when $n = 1$ since the subcollection is empty in that case.) Furthermore, if $x = x_0$ and $y = y_0$ are zero lines of f of multiplicity n, then in addition to the above,

$$(\partial_x^n \partial_y^n f)(x_0, y_0) = (\partial_x^{n-1} \partial_y^{n-1})(\partial_x \partial_y f)(x_0, y_0) \neq 0.$$

It follows that $x = x_0$ and $y = y_0$ are zero lines of $\partial_x \partial_y f$ of multiplicity $n-1$. This property of zero lines will help us prove the following bivariate analogue of Proposition 8.2:

Proposition 8.23 *Let $X, Y \subset \mathbb{R}$ be intervals, and let $x_0 \in X$ and $y_0 \in Y$. Assume that $f \in C^{(n,n)}(X \times Y)$, where $n \in \mathbb{N}$ is arbitrary. If $x = x_0$ and $y = y_0$ are zero lines of f of multiplicity n, then $\{\partial_x^i \partial_y^i f\}_{i=0}^n \subset \mathrm{LNV}_{(x_0,y_0)}(X \times Y)$.*

Proof. This simply *adorable* proof skips winsomely along by induction on n. Let P_n denote the proposition under consideration.

We must first prove proposition P_0. Assume that $f \in C(X \times Y)$. If $x = x_0$ and $y = y_0$ are zero lines of f of multiplicity 0, then $f(x_0, y_0) \neq 0$. By the continuity of f at (x_0, y_0), there is an open neighborhood of (x_0, y_0) in which f does not vanish. This shows that $f \in \mathrm{LNV}_{(x_0,y_0)}(X \times Y)$, and thus proposition P_0 holds.

Now suppose that proposition P_n holds for some $n \in \mathbb{N}$. We must show that proposition P_{n+1} follows. Assume that $f \in C^{(n+1,n+1)}(X \times Y)$. *Here comes the adorable part!* If $x = x_0$ and $y = y_0$ are zero lines of f of multiplicity $n+1$, then $x = x_0$ and $y = y_0$ are zero lines of $\partial_x \partial_y f \in C^{(n,n)}(X \times Y)$ of multiplicity n. The induction hypothesis implies that

$$\{(\partial_x^i \partial_y^i)(\partial_x \partial_y f)\}_{i=0}^n = \{\partial_x^i \partial_y^i f\}_{i=1}^{n+1} \subset \mathrm{LNV}_{(x_0,y_0)}(X \times Y).$$

All we need to show is that $f \in \mathrm{LNV}_{(x_0,y_0)}(X \times Y)$. Since $\partial_x \partial_y f \in \mathrm{LNV}_{(x_0,y_0)}(X \times Y)$, there exist open neighborhoods M and N of x_0 and y_0, respectively, such that $\partial_x \partial_y f$ does not vanish on the set

$$S := [(X \cap M) \backslash \{x_0\}] \times [(Y \cap N) \backslash \{y_0\}].$$

Note that S has at most four connected components (less than four if x_0 is an endpoint of the interval X or y_0 is an endpoint of the interval Y). Each one of these connected components is a rectangle with the point (x_0, y_0) as one vertex.

We will now prove by contradiction that f cannot vanish on S. Suppose that $(x_1, y_1) \in S$ is a zero of f. Since the lines $x = x_0$ and $y = y_0$ do not intersect the set S, the point (x_1, y_1) does not lie on either of these lines. Since $x = x_0$ and $y = y_0$ are zero lines of f by assumption, Corollary 8.21 implies that $\partial_x \partial_y f$ must vanish at some point (ξ, η) which lies in the open rectangle R with vertices $\{(x_i, y_j) : 0 \leq i, j \leq 1\}$. Let S_1 denote the connected component of S containing the point (x_1, y_1). Since S_1 is a rectangle with vertex (x_0, y_0), and since $(x_1, y_1) \in S_1$, the geometry of the situation implies that the open rectangle R is completely contained in the rectangle S_1 (see Figure 8.1). In summary,

$$(\xi, \eta) \in R \subset S_1 \subset S.$$

This implies that $(\xi, \eta) \in S$, which contradicts that $\partial_x \partial_y f$ does not vanish on S. We conclude that f has no zeros in S. This shows that $f \in \mathrm{LNV}_{(x_0,y_0)}(X \times Y)$, and thus proposition P_{n+1} holds.

We have shown that proposition P_0 holds, and that proposition P_n implies proposition P_{n+1} whenever $n \in \mathbb{N}$. By induction on n, proposition P_n holds for all $n \in \mathbb{N}$. ■

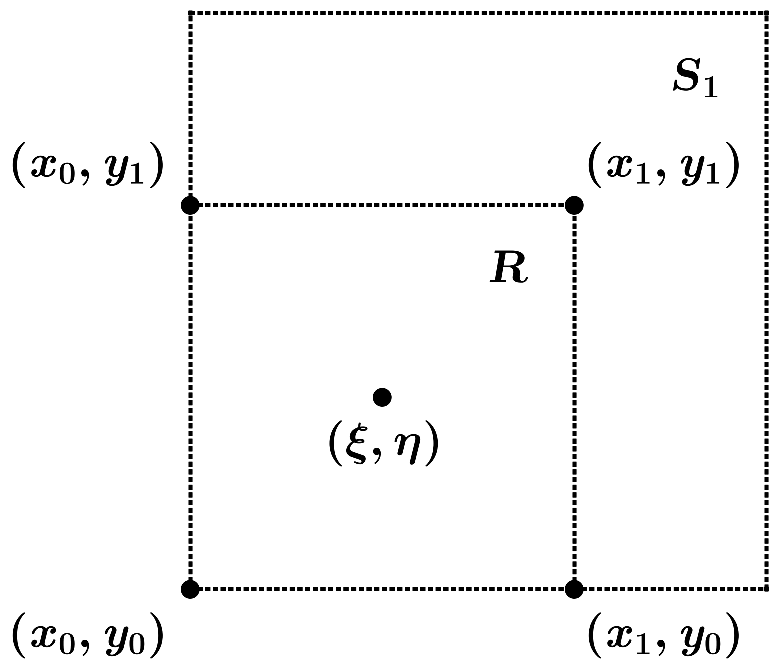

Figure 8.1: The Geometry of the Proof of Proposition 8.23

We will now define the bivariate analogues of pre-l'Hôpital and l'Hôpital sequences of smooth functions of two real variables:

Definition 8.24 *Let $X, Y \subset \mathbb{R}$ be intervals, let $x_0 \in X$ and $y_0 \in Y$, and let n denote either a positive integer or infinity. Let $\{r_i\}_{i=0}^{n} \subset C^{(n-1,n-1)}(X \times Y)$ be an arbitrary sequence. If $x = x_0$ and $y = y_0$ are zero lines of r_i of multiplicity at least i for $0 \le i < n+1$, we say that the sequence $\{r_i\}_{i=0}^{n}$* ***is pre-l'Hôpital at*** *(x_0, y_0). More explicitly, this means that*

$$E_{x_0} \partial_x^i r_j = 0 \quad \textit{and} \quad E^{y_0} \partial_y^i r_j = 0 \quad \textit{for} \quad 0 \le i < j < n+1.$$

Note that this condition asserts nothing about the function r_0.

Now let $\{r_i\}_{i=0}^{n-1} \subset C^{(n-1,n-1)}(X \times Y)$ be an arbitrary sequence. If $x = x_0$ and $y = y_0$ are zero lines of r_i of multiplicity i for $0 \le i < n$, we say that the sequence $\{r_i\}_{i=0}^{n-1}$ ***is***

l'Hôpital at (x_0, y_0). *More explicitly, this means that*

$$E_{x_0}\partial_x^i r_j = 0 \quad \textit{and} \quad E^{y_0}\partial_y^i r_j = 0 \quad \textit{for} \quad 0 \le i < j < n$$

and

$$(\partial_x^i \partial_y^i r_i)(x_0, y_0) \neq 0 \quad \textit{for} \quad 0 \le i < n.$$

The previous definition motivates a further development of the logical parametric extensions and the logical tensor product which we defined earlier for propositions about *individual functions:* We can also define these constructions for propositions about *sequences of functions*. For example, assume that $\{u_i\}_{i=0}^n \subset C^{n-1}(X)$, that $\{v_i\}_{i=0}^n \subset C^{n-1}(Y)$, and that $\{r_i\}_{i=0}^n \subset C^{(n-1,n-1)}(X \times Y)$. Let $P(x_0; \{u_i\}_{i=0}^n)$ denote the proposition that the sequence of functions $\{u_i\}_{i=0}^n$ is pre-l'Hôpital at x_0. We define the logical parametric extension of this proposition, denoted by $\tilde{P}(x_0; \{r_i\}_{i=0}^n)$, to be the proposition which asserts that $P(x_0; \{(r_i)^y\}_{i=0}^n)$ holds for all $y \in Y$. The resulting proposition asserts that for each fixed $y \in Y$, the sequence of y-sections $\{(r_i)^y\}_{i=0}^n \subset C^{n-1}(X)$ is pre-l'Hôpital at x_0. This means that $x = x_0$ is a zero line of r_i of multiplicity at least i for $0 \le i < n+1$. Similarly, if $Q(y_0; \{v_i\}_{i=0}^n)$ denotes the proposition that the sequence of functions $\{v_i\}_{i=0}^n$ is pre-l'Hôpital at y_0, the logical parametric extension $\tilde{Q}(y_0; \{r_i\}_{i=0}^n)$ asserts that $Q(y_0; \{(r_i)_x\}_{i=0}^n)$ holds for all $x \in X$. This means that $y = y_0$ is a zero line of r_i of multiplicity at least i for $0 \le i < n + 1$. We define the logical tensor product $(\tilde{P} \otimes \tilde{Q})(x_0, y_0; \{r_i\}_{i=0}^n)$ to be the proposition which asserts that both $\tilde{P}(x_0; \{r_i\}_{i=0}^n)$ and $\tilde{Q}(y_0; \{r_i\}_{i=0}^n)$ hold simultaneously. This means that $x = x_0$ and $y = y_0$ are zero lines of r_i of multiplicity at least i for $0 \le i < n + 1$—in other words, it means that $\{r_i\}_{i=0}^n$ is pre-l'Hôpital at (x_0, y_0). We summarize these observations in the following remark:

Remark 8.25 *The bivariate proposition which asserts that the sequence of functions* $\{r_i\}_{i=0}^n \subset C^{(n-1,n-1)}(X \times Y)$ *is pre-l'Hôpital at* (x_0, y_0) *has a special logical structure: It is the logical tensor product of the logical parametric extensions of two univariate propositions—the proposition which asserts that* $\{u_i\}_{i=0}^n \subset C^{n-1}(X)$ *is pre-l'Hôpital at* x_0 *and the proposition which asserts that* $\{v_i\}_{i=0}^n \subset C^{n-1}(Y)$ *is pre-l'Hôpital at* y_0. *In the special case where* $r_i = f$ *for* $0 \le i < n + 1$, *this observation reduces to the observation of Remark 8.22.*

The basic proof technique illustrated by Logical Tensor Product Implication Theorem 8.16 and Logical Tensor Product Equivalence Corollary 8.17 can be extended in a straightforward fashion by replacing propositions about individual functions with propositions about sequences of functions. We will call upon these extensions in the next subsection.

The following bivariate analogue of Proposition 8.6 follows immediately from Proposition 8.23:

Proposition 8.26 *Let $X, Y \subset \mathbb{R}$ be intervals, let $x_0 \in X$ and $y_0 \in Y$, and let n denote either a positive integer or infinity. If $\{r_i\}_{i=0}^{n-1} \subset C^{(n-1,n-1)}(X \times Y)$ is l'Hôpital at (x_0, y_0), then $\{r_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{(x_0,y_0)}(X \times Y)$.*

What is the bivariate analogue of Taylor interpolation? Let $f_1, f_2 \in C^{(n,n)}(X \times Y)$, where $n \in \mathbb{N}$ is arbitrary. We say that f_1 **Taylor interpolates** f_2 **to order** n **on** $x = x_0$ **and** $y = y_0$ if the higher-order normal derivatives of f_1 and f_2 satisfy

$$E_{x_0}\partial_x^i f_1 = E_{x_0}\partial_x^i f_2 \quad \text{and} \quad E^{y_0}\partial_y^i f_1 = E^{y_0}\partial_y^i f_2 \quad \text{for} \quad 0 \le i \le n.$$

This is clearly an equivalence relation on $C^{(n,n)}(X \times Y)$. Note that f_1 Taylor interpolates f_2 to order n on $x = x_0$ and $y = y_0$ if and only if $x = x_0$ and $y = y_0$ are zero lines of $f_1 - f_2$ of multiplicity at least $n + 1$; this equivalent reformulation is sometimes more convenient.

Remark 8.27 *Assume that $u_i \in C^n(X)$, that $v_i \in C^n(Y)$, and that $f_i \in C^{(n,n)}(X \times Y)$ for $i = 1, 2$. The bivariate proposition which asserts that f_1 Taylor interpolates f_2 to order n on $x = x_0$ and $y = y_0$ has a special logical structure: It is the logical tensor product of the logical parametric extensions of two univariate propositions—the proposition which asserts that u_1 Taylor interpolates u_2 to order n at x_0 and the proposition which asserts that v_1 Taylor interpolates v_2 to order n at y_0. This fact comes sharply into focus if we reformulate the Taylor interpolation of f_1 by f_2 in terms of the zero lines of the difference $f_1 - f_2$ and apply Remark 8.22.*

The primary goal of this chapter is to perform Taylor interpolation on $x = x_0$ and $y = y_0$ using dual asymptotic expansions. In order to achieve this goal, we must first define what it means for a dual asymptotic series and a dual asymptotic expansion to be l'Hôpital at (x_0, y_0):

Definition 8.28 *Let $X, Y \subset \mathbb{R}$ be intervals, and let $x_0 \in X$ and $y_0 \in Y$. Assume that $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}\backslash\{0\}$, where n is either a positive integer or infinity. If the sequence $\{g_i\}_{i=0}^{n-1} \subset C^n(X)$ is l'Hôpital at x_0 and the sequence $\{h_i\}_{i=0}^{n-1} \subset C^n(Y)$ is l'Hôpital at y_0, we say that the dual asymptotic series*

$$s_n := \sum_{i=0}^{n-1} c_i \cdot (g_i \otimes h_i)$$

is l'Hôpital at (x_0, y_0). *Assume that* $f \in C^{(n,n)}(X \times Y)$. *If the dual asymptotic expansion*

$$f(x,y) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x)\, h_i(y) \quad as \quad x \to x_0 \quad or \quad y \to y_0$$

holds under these hypotheses, we say that the dual asymptotic expansion ***is l'Hôpital at*** (x_0, y_0). *We also call this series a* ***l'Hôpital dual asymptotic expansion of*** f ***to*** n ***terms at*** (x_0, y_0). *In addition, if* $\{g_i\}_{i=0}^{n-1}$ *is point-normalized at* x_0 *and* $\{h_i\}_{i=0}^{n-1}$ *is point-normalized at* y_0, *we say that the dual asymptotic series and the l'Hôpital dual asymptotic expansion* ***are point-normalized at*** (x_0, y_0). *We denote the set of all functions in* $C^{(n,n)}(X \times Y)$ *which have l'Hôpital dual asymptotic expansions to* n *terms at* (x_0, y_0) *by* $\mathrm{LDAE}^n_{(x_0,y_0)}(X \times Y)$.

Why does a l'Hôpital dual asymptotic expansion to n terms at (x_0, y_0) possess more smoothness than is necessary to consider Taylor interpolation to order $n-1$ on the lines $x = x_0$ and $y = y_0$? We impose this additional smoothness in anticipation of the needs of the *remainder theory* that we will develop for Taylor interpolation later in this chapter.

What is the logical structure of the bivariate proposition which defines a l'Hôpital dual asymptotic expansion? The following remark answers this question:

Remark 8.29 *As we noted earlier in Example 8.13, the bivariate proposition which defines a generic dual asymptotic expansion at* (x_0, y_0) *has a special structure: It is the logical tensor product of the logical parametric extensions of two univariate propositions which define univariate asymptotic expansions at* x_0 *and* y_0, *modulo suitable parameters. These parameters include the asymptotic sequence* $\{g_i(x)\}_{i=0}^{n-1}$ *as* $x \to x_0$ *and the asymptotic sequence* $\{h_i(y)\}_{i=0}^{n-1}$ *as* $y \to y_0$. *If we replace these generic asymptotic sequences by sequences* $\{g_i\}_{i=0}^{n-1}$ *and* $\{h_i\}_{i=0}^{n-1}$ *which are l'Hôpital at* x_0 *and* y_0, *respectively, the underlying logical structure is preserved. We conclude that the bivariate proposition which defines a l'Hôpital dual asymptotic expansion at* (x_0, y_0) *also has a special structure: It is the logical tensor product of the logical parametric extensions of two univariate propositions which define univariate asymptotic expansions which are l'Hôpital at* x_0 *and* y_0, *modulo suitable parameters.*

Let $\{s_i\}_{i=0}^{n}$ denote the sequence of partial sums of the l'Hôpital dual asymptotic expansion above. As in the univariate case, if $n < \infty$, the dual asymptotic series s_n is a finite sum and satisfies $s_n \in C^{(n,n)}(X \times Y)$; in that case, it is clear what we mean if we say that s_n Taylor interpolates f to order $n-1$ on $x = x_0$ and $y = y_0$. If $n = \infty$, the partial sums of the formal infinite series s_∞ satisfy $\{s_i\}_{i=0}^{\infty} \subset C^{(\infty,\infty)}(X \times Y)$; in that case, we say that s_∞ **Taylor interpolates** f **to order** ∞ **on** $x = x_0$ **and** $y = y_0$ if s_i Taylor interpolates f to order $i-1$ on $x = x_0$ and $y = y_0$ for all $i \geq 1$.

Remark 8.30 *The definition of Taylor interpolation by a formal infinite series in two variables to order ∞ on $x = x_0$ and $y = y_0$ has the familiar logical structure: It is the logical tensor product of logical parametric extensions of the definitions of Taylor interpolation by formal infinite series in one variable to order ∞ at x_0 and y_0.*

We have now established all of the definitions and fundamental results we will need to develop the theory of l'Hôpital dual asymptotic expansions in the next subsection.

8.3.2 The Theory of l'Hôpital Dual Asymptotic Expansions

In this subsection, we will develop a specialized theory for l'Hôpital dual asymptotic expansions. We will begin by proving a bivariate analogue of Univariate Taylor Interpolation Theorem 8.8. We will then use this bivariate Taylor interpolation theorem to develop a uniqueness theorem and an existence theorem for l'Hôpital dual asymptotic expansions. We will conclude this subsection by using these uniqueness and existence theorems to develop an iterative method which decides the question of existence algorithmically.

The following theorem explains how to perform Taylor interpolation on $x = x_0$ and $y = y_0$ using dual asymptotic series which are both l'Hôpital and point-normalized at (x_0, y_0):

Theorem 8.31 (Bivariate Taylor Interpolation) *Let $X, Y \subset \mathbb{R}$ be intervals, let $x_0 \in X$ and $y_0 \in Y$, and let $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}\backslash\{0\}$, where n is either a positive integer or infinity. Assume that $f \in C^{(n,n)}(X \times Y)$, that the sequence $\{g_i\}_{i=0}^{n-1} \subset C^n(X)$ is both l'Hôpital and point-normalized at x_0, and that the sequence $\{h_i\}_{i=0}^{n-1} \subset C^n(Y)$ is both l'Hôpital and point-normalized at y_0. The following four statements are equivalent:*

1. *The point-normalized l'Hôpital dual asymptotic series*

$$s_n := \sum_{i=0}^{n-1} c_i \cdot (g_i \otimes h_i)$$

Taylor interpolates f to order $n-1$ on $x = x_0$ and $y = y_0$.

2. *The coefficient functions* $\{c_i \cdot h_i\}_{i=0}^{n-1}$ *and* $\{c_i \cdot g_i\}_{i=0}^{n-1}$ *can be calculated recursively from the remainders* $\{r_i\}_{i=0}^{n}$ *via the following equations, which are valid for* $0 \le i < n$*:*

$$c_i \cdot h_i = E_{x_0} \partial_x^i r_i \tag{8.3a}$$

$$c_i \cdot g_i = E^{y_0} \partial_y^i r_i. \tag{8.3b}$$

3. *The sequence of remainders* $\{r_i\}_{i=0}^{n}$ *is pre-l'Hôpital at* (x_0, y_0)*.*

4. *The following point-normalized l'Hôpital dual asymptotic expansion holds:*

$$f(x,y) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x)\, h_i(y) \quad \text{as} \quad x \to x_0 \quad \text{or} \quad y \to y_0.$$

Proof. Let us fix an arbitrary value $y \in Y$. If we use Univariate Taylor Interpolation Theorem 8.8 to calculate the coefficients $\{c_i \cdot h_i(y)\}_{i=0}^{n-1}$ of the y-sections of the l'Hôpital dual asymptotic expansion above with respect to the asymptotic sequence $\{g_i(x)\}_{i=0}^{n-1}$ as $x \to x_0$, we obtain

$$c_i \cdot h_i(y) = D^i(r_i)^y(x_0) = \partial_x^i r_i(x_0, y) \quad \text{for} \quad 0 \le i < n.$$

Since this equation holds for all $y \in Y$, we can rewrite this in terms of higher-order normal derivative operators as equation (8.3a). Similarly, for each fixed $x \in X$, using Univariate Taylor Interpolation Theorem 8.8 to calculate the coefficients $\{c_i \cdot g_i(x)\}_{i=0}^{n-1}$ of the x-sections of the l'Hôpital dual asymptotic expansion above with respect to the asymptotic sequence $\{h_i(y)\}_{i=0}^{n-1}$ as $y \to y_0$ ultimately yields equation (8.3b).

The bivariate proposition expressed by equation (8.3) is the logical tensor product of the logical parametric extensions of a pair of univariate propositions which are based on equation (8.1) of Univariate Taylor Interpolation Theorem 8.8, modulo suitable parameters. This observation, together with the observations of Remarks 8.25, 8.27, 8.29, and 8.30, allow us to conclude that all four bivariate propositions listed in Bivariate Taylor Interpolation Theorem 8.31 have the same logical structure: They are all logical tensor products of the logical parametric extensions of pairs of univariate propositions which are based on the corresponding four univariate propositions in Univariate Taylor Interpolation Theorem 8.8. Since Univariate Taylor Interpolation Theorem 8.8 asserts that these four univariate propositions are equivalent, and since $(C^n(X),\ C^n(Y),\ C^{(n,n)}(X \times Y))$ is a cross-sectionally compatible triple, we conclude by a straightforward extension of Logical

Tensor Product Equivalence Corollary 8.17 that the four bivariate propositions listed in Bivariate Taylor Interpolation Theorem 8.31 are equivalent as well. ■

The previous theorem states that an n-term dual asymptotic series which is both l'Hôpital and point-normalized at (x_0, y_0) will Taylor interpolate a function f to order $n-1$ on $x = x_0$ and $y = y_0$ if and only if the l'Hôpital dual asymptotic series is a l'Hôpital dual asymptotic expansion of f to n terms at (x_0, y_0). The theorem also provides an equivalent condition that tells us how to calculate the coefficient functions $\{c_i \cdot h_i\}_{i=0}^{n-1}$ and $\{c_i \cdot g_i\}_{i=0}^{n-1}$ of the l'Hôpital dual asymptotic expansion via the higher-order normal derivatives of the remainders $\{r_i\}_{i=0}^{n-1}$ on the lines $x = x_0$ and $y = y_0$. We will soon take this a step further by developing a method for calculating the individual coefficients $\{c_i\}_{i=0}^{n-1}$ and the univariate functions $\{g_i\}_{i=0}^{n-1}$ and $\{h_i\}_{i=0}^{n-1}$ separately.

Definition 8.28 implies that

$$\mathrm{LDAE}^n_{(x_0,y_0)}(X \times Y) \subset C^{(n,n)}(X \times Y).$$

We will now show that every *l'Hôpital* dual asymptotic expansion is also a *weak* dual asymptotic expansion, which will imply that

$$\mathrm{LDAE}^n_{(x_0,y_0)}(X \times Y) \subset \mathrm{WDAE}^{\,n}_{(x_0,y_0)}(X \times Y) \cap C^{(n,n)}(X \times Y).$$

We will obtain this result in the course of proving the following *uniqueness theorem for l'Hôpital dual asymptotic expansions:*

Theorem 8.32 (Uniqueness) *Let $X, Y \subset \mathbb{R}$ be intervals, let $x_0 \in X$ and $y_0 \in Y$, and let n be either a positive integer or infinity. If $f \in \mathrm{LDAE}^n_{(x_0,y_0)}(X \times Y)$ has the following l'Hôpital dual asymptotic expansion*

$$f(x, y) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x)\, h_i(y) \quad \textit{as} \quad x \to x_0 \quad \textit{or} \quad y \to y_0 \tag{8.4}$$

with remainders $\{r_i\}_{i=0}^{n}$, then this expansion is also a weak dual asymptotic expansion, and thus $f \in \mathrm{WDAE}^{\,n}_{(x_0,y_0)}(X \times Y)$. Uniqueness Theorem 7.27 further implies that the terms

$$\{c_i \cdot (g_i \otimes h_i)\}_{i=0}^{n-1} \subset C^{(n,n)}(X \times Y)$$

of this expansion are uniquely determined by the function $f \in C^{(n,n)}(X \times Y)$ and the point $(x_0, y_0) \in X \times Y$ alone. In addition, if this l'Hôpital dual asymptotic expansion *is point-normalized at* (x_0, y_0)*, then the coefficients* $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}\backslash\{0\}$ *and the functions* $\{g_i\}_{i=0}^{n-1} \subset C^n(X)$ *and* $\{h_i\}_{i=0}^{n-1} \subset C^n(Y)$ *are all uniquely determined as well, and can be calculated recursively from the remainders* $\{r_i\}_{i=0}^{n} \subset C^{(n,n)}(X \times Y)$ *via the following equations, which are valid for* $0 \leq i < n$*:*

$$\begin{aligned} c_i &= (\partial_x^i \partial_y^i r_i)(x_0, y_0) && \text{(8.5a)} \\ g_i &= (E^{y_0} \partial_y^i r_i)/c_i && \text{(8.5b)} \\ h_i &= (E_{x_0} \partial_x^i r_i)/c_i. && \text{(8.5c)} \end{aligned}$$

Proof. We can always rewrite the terms of the given l'Hôpital dual asymptotic expansion (8.4) to obtain a l'Hôpital dual asymptotic expansion

$$f(x,y) \sim \sum_{i=0}^{n-1} c_i\, D^i g_i(x_0)\, D^i h_i(y_0) \cdot \frac{g_i(x)}{D^i g_i(x_0)} \frac{h_i(y)}{D^i h_i(y_0)} \quad \text{as} \quad x \to x_0 \quad \text{or} \quad y \to y_0$$

which is point-normalized at (x_0, y_0); consequently, we can assume without loss of generality that the given l'Hôpital dual asymptotic expansion (8.4) is already point-normalized at (x_0, y_0).

By Definition 8.28, the sequence $\{g_i\}_{i=0}^{n-1} \subset C^n(X)$ is both l'Hôpital and point-normalized at x_0, and the sequence $\{h_i\}_{i=0}^{n-1} \subset C^n(Y)$ is both l'Hôpital and point-normalized at y_0. It follows by Proposition 8.6 that $\{g_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{x_0}(X)$ and $\{h_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{y_0}(Y)$. According to Definition 7.22, all that remains is to show that

$$\{r_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{(x_0,y_0)}(X \times Y) \cup \mathrm{LNV}_{x_0}^{y_0}(X \times Y).$$

By Proposition 8.26, it will suffice to prove that the sequence $\{r_i\}_{i=0}^{n-1} \subset C^{(n,n)}(X \times Y)$ is l'Hôpital at (x_0, y_0) since this implies that $\{r_i\}_{i=0}^{n-1} \subset \mathrm{LNV}_{(x_0,y_0)}(X \times Y)$.

Since l'Hôpital dual asymptotic expansion (8.4) holds by hypothesis, Bivariate Taylor Interpolation Theorem 8.31 implies that the sequence of remainders $\{r_i\}_{i=0}^{n}$ is pre-l'Hôpital at (x_0, y_0). It follows immediately that the subsequence $\{r_i\}_{i=0}^{n-1}$ is also pre-l'Hôpital at (x_0, y_0). In order to show that $\{r_i\}_{i=0}^{n-1}$ is l'Hôpital at (x_0, y_0), it now suffices to prove that

$$(\partial_x^i \partial_y^i r_i)(x_0, y_0) \neq 0 \quad \text{for} \quad 0 \leq i < n. \tag{8.6}$$

Bivariate Taylor Interpolation Theorem 8.31 implies that

$$c_i \cdot g_i = E^{y_0} \partial_y^i r_i \quad \text{for} \quad 0 \leq i < n.$$

Differentiating the previous equation i times and evaluating the result at x_0 yields

$$c_i \cdot D^i g_i(x_0) = E_{x_0} E^{y_0} (\partial_x^i \partial_y^i r_i) = (\partial_x^i \partial_y^i r_i)(x_0, y_0) \quad \text{for} \quad 0 \leq i < n.$$

Since by hypothesis the sequence $\{g_i\}_{i=0}^{n-1}$ is point-normalized at x_0, it follows that $D^i g_i(x_0) = 1$ for $0 \leq i < n$, and thus the previous equation simplifies to equation (8.5a). By Remark 7.21, we *assume* that the coefficient $c_i \neq 0$ for $0 \leq i < n$. Since the left-hand side of equation (8.5a) is nonzero, it follows that equation (8.6) holds. This completes the proof that $\{r_i\}_{i=0}^{n-1}$ is l'Hôpital at (x_0, y_0).

We have now proven that l'Hôpital dual asymptotic expansion (8.4) is a weak dual asymptotic expansion, and thus $f \in \mathrm{WDAE}^n_{(x_0,y_0)}(X \times Y)$. By Uniqueness Theorem 7.27, the terms $\{c_i \cdot (g_i \otimes h_i)\}_{i=0}^{n-1}$ of this weak dual asymptotic expansion are uniquely determined, which implies that the partial sums $\{s_i\}_{i=0}^{n}$ and the remainders $\{r_i\}_{i=0}^{n}$ are also uniquely determined. It follows that the coefficients $\{c_i\}_{i=0}^{n-1}$ are uniquely determined by equation (8.5a), which we established earlier. Dividing equation (8.3) of Bivariate Taylor Interpolation Theorem 8.31 by the coefficient $c_i \neq 0$ establishes equations (8.5b) and (8.5c), which uniquely determine the functions $\{g_i\}_{i=0}^{n-1}$ and $\{h_i\}_{i=0}^{n-1}$ as well. ∎

Uniqueness Theorem 8.32 tells us that a point-normalized l'Hôpital dual asymptotic expansion (8.4) enjoys an even stronger kind of uniqueness than a generic weak dual asymptotic expansion—for not only are the terms $\{c_i \cdot (g_i \otimes h_i)\}_{i=0}^{n-1}$ uniquely determined by the function f and the point (x_0, y_0) alone, but the coefficients $\{c_i\}_{i=0}^{n-1}$ and the functions $\{g_i\}_{i=0}^{n-1}$ and $\{h_i\}_{i=0}^{n-1}$ are uniquely determined as well. Furthermore, instead of calculating these terms using the original limiting operations that define the asymptotic splitting operator $\Upsilon_{(x_0 \cdot y_0)}$, we can calculate these terms using another kind of limiting operation: normal differentiation on the lines $x = x_0$ and $y = y_0$. The advantage of the latter approach is that we can calculate normal derivatives using an algebra of well-established symbolic differentiation rules instead of evaluating the underlying iterated one-dimensional limits directly. This advantage demonstrates the value of the l'Hôpital hypothesis:

Remark 8.33 *The l'Hôpital hypothesis for dual asymptotic expansions allows us to use l'Hôpital's rule in a systematic way to reduce the evaluation of the one-dimensional limits in the asymptotic splitting operator to the evaluation of higher-order normal derivatives.*

We can exploit the above observation to develop another important special case of the *asymptotic* splitting operator; this, in turn, provides another sneak preview of the *abstract* splitting operator that is to come. Uniqueness Theorems 7.27 and 8.32 together imply that the functions $\{\Upsilon_{(x_0,y_0)}\, r_i\}_{i=0}^{n-1}$ are well-defined and have the following special form:

$$\Upsilon_{(x_0,y_0)}\, r_i = \frac{(E^{y_0}\partial_y^i r_i) \otimes (E_{x_0}\partial_x^i r_i)}{(E_{x_0}\partial_x^i)(E^{y_0}\partial_y^i) r_i} \quad \text{for} \quad 0 \leq i < n.$$

We can simplify this further by letting $\Phi_i := E_{x_0}\partial_x^i$ and $\Psi_i := E^{y_0}\partial_y^i$ denote the parametric extensions of the linear functionals $\phi_i := \varepsilon_{x_0} D_x^i$ and $\psi_i := \varepsilon^{y_0} D_y^i$, following the general notational conventions of Section 3.2. In the i-th iteration, we can now rewrite the *asymptotic* splitting operator $\Upsilon_{(x_0,y_0)}$ at the point (x_0, y_0) as an *abstract* splitting operator $\Omega_{(\phi_i,\psi_i)}$ induced by the pair of linear functionals (ϕ_i, ψ_i); this final transformation yields

$$\Omega_{(\phi_i,\psi_i)}\, r_i = \frac{\Psi_i\, r_i \otimes \Phi_i\, r_i}{\Phi_i \Psi_i\, r_i} \quad \text{for} \quad 0 \leq i < n.$$

Remark 8.34 *The reformulation of the asymptotic splitting operator $\Upsilon_{(x_0,y_0)}$ in the i-th iteration as the abstract splitting operator $\Omega_{(\phi_i,\psi_i)}$ reveals that l'Hôpital dual asymptotic expansions are yet another example of the more general class of series expansions which the author calls Geddes series.*

We will now prove the following *sufficient condition for the existence of a point-normalized l'Hôpital dual asymptotic expansion:*

Theorem 8.35 (Existence) *Let $X, Y \subset \mathbb{R}$ be intervals, let $x_0 \in X$ and $y_0 \in Y$, and assume that $f \in C^{(n,n)}(X \times Y)$. Let $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}$ and $\{g_i\}_{i=0}^{n-1} \subset C^n(X)$ and $\{h_i\}_{i=0}^{n-1} \subset C^n(Y)$, where n is either a positive integer or ∞. Let $\{r_i\}_{i=0}^{n-1}$ denote the sequence of remainders of the function f with respect to the formal series*

$$\sum_{i=0}^{n-1} c_i \cdot (g_i \otimes h_i),$$

and note that $\{r_i\}_{i=0}^{n-1} \subset C^{(n,n)}(X \times Y)$. If the following three-part condition holds for

$0 \le i < n$,

$$c_i = (\partial_x^i \partial_y^i r_i)(x_0, y_0) \neq 0 \tag{8.7a}$$
$$g_i = (E^{y_0} \partial_y^i r_i)/c_i \tag{8.7b}$$
$$h_i = (E_{x_0} \partial_x^i r_i)/c_i, \tag{8.7c}$$

then the sequence $\{g_i\}_{i=0}^{n-1}$ is both l'Hôpital and point-normalized at x_0, the sequence $\{h_i\}_{i=0}^{n-1}$ is both l'Hôpital and point-normalized at y_0, and the following point-normalized l'Hôpital dual asymptotic expansion holds:

$$f(x, y) \sim \sum_{i=0}^{n-1} c_i \cdot g_i(x)\, h_i(y) \quad \textit{as} \quad x \to x_0 \quad \textit{or} \quad y \to y_0. \tag{8.8}$$

Proof. By the definition of the $n = \infty$ case, it suffices to prove that the theorem is true for all positive integers n. Let P_n denote the logical proposition asserted by the theorem. The proof will proceed by induction on n.

We must first show that proposition P_1 holds. Assume that $f \in C^{(1,1)}(X \times Y)$, let $c_0 \in \mathbb{R}$ and $g_0 \in C^1(X)$ and $h_0 \in C^1(Y)$, and define $r_0 := f$. Condition (8.7) for $i = 0$ states that $c_0 = r_0(x_0, y_0) \neq 0$, that $g_0 = (E^{y_0} r_0)/c_0$, and that $h_0 = (E_{x_0} r_0)/c_0$. Since

$$g_0(x_0) = h_0(y_0) = r_0(x_0, y_0)/c_0 = 1,$$

the one-term sequences $\{g_0\}$ and $\{h_0\}$ are both l'Hôpital and point-normalized at x_0 and y_0, respectively. Since the coefficient functions satisfy $c_0 \cdot g_0 = E^{y_0} r_0$ and $c_0 \cdot h_0 = E_{x_0} r_0$, Bivariate Taylor Interpolation Theorem 8.31 for $n = 1$ implies that $s_1 := c_0 \cdot (g_0 \otimes h_0)$ is a point-normalized l'Hôpital dual asymptotic expansion of f to one term at (x_0, y_0). This proves that proposition P_1 holds.

Suppose that proposition P_n holds for an arbitrary positive integer n. We must show that proposition P_{n+1} follows. Assume the hypotheses of proposition P_{n+1}, and note that these imply the hypotheses of proposition P_n. Condition (8.7) for $i = n$ states that $c_n = (\partial_x^n \partial_y^n r_n)(x_0, y_0) \neq 0$, that $g_n = (E^{y_0} \partial_y^n r_n)/c_n$, and that $h_n = (E_{x_0} \partial_x^n r_n)/c_n$.

Since proposition P_n implies that point-normalized l'Hôpital dual asymptotic expansion (8.8) holds, it follows from Bivariate Taylor Interpolation Theorem 8.31 that the sequence of remainders $\{r_i\}_{i=0}^{n}$ is pre-l'Hôpital at (x_0, y_0). This implies that $x = x_0$ and $y = y_0$ are zero lines of r_n of multiplicity at least n, which means that the following univariate identities hold:

$$\partial_x^i r_n(x_0, y) \equiv 0 \quad \text{and} \quad \partial_y^i r_n(x, y_0) \equiv 0 \quad \text{for} \quad 0 \le i < n.$$

We can differentiate these identities n times each to obtain the new identities

$$(\partial_y^n \partial_x^i r_n)(x_0, y) \equiv 0 \quad \text{and} \quad (\partial_x^n \partial_y^i r_n)(x, y_0) \equiv 0 \quad \text{for} \quad 0 \le i < n.$$

Substituting $y = y_0$ and $x = x_0$ into these new identities yields the following collection of n equations:

$$(\partial_y^n \partial_x^i r_n)(x_0, y_0) = (\partial_x^n \partial_y^i r_n)(x_0, y_0) = 0 \quad \text{for} \quad 0 \le i < n.$$

Since the partial derivative operators commute, we can rewrite these equations in terms of the univariate coefficient functions $c_n \cdot g_n$ and $c_n \cdot h_n$ to obtain

$$c_n \cdot D^i g_n(x_0) = c_n \cdot D^i h_n(y_0) = 0 \quad \text{for} \quad 0 \le i < n.$$

Since $c_n \ne 0$ by assumption, we can cancel the constant factor c_n to obtain

$$D^i g_n(x_0) = D^i h_n(y_0) = 0 \quad \text{for} \quad 0 \le i < n.$$

The previous collection of n equations and the following equation

$$D^n g_n(x_0) = D^n h_n(y_0) = (\partial_x^n \partial_y^n r_n)(x_0, y_0)/c_0 = 1$$

together imply that x_0 is a zero of g_n of multiplicity n and that y_0 is a zero of h_n of multiplicity n as well. Since Proposition P_n implies that the sequences $\{g_i\}_{i=0}^{n-1}$ and $\{h_i\}_{i=0}^{n-1}$ are both l'Hôpital and point-normalized at x_0 and y_0, respectively, we conclude that the extended sequences $\{g_i\}_{i=0}^{n}$ $\{h_i\}_{i=0}^{n}$ are also both l'Hôpital and point-normalized at x_0 and y_0, respectively.

We have now deduced the first two of the three conclusions of proposition P_{n+1}. This in turn establishes that all the general hypotheses of Bivariate Taylor Interpolation Theorem 8.31 are satisfied with n replaced by $n+1$. Since condition (8.7) implies that the coefficient functions satisfy $c_i \cdot g_i = E^{y_0} \partial_y^i r_i$ and $c_i \cdot h_i = E_{x_0} \partial_x^i r_i$ for $0 \le i < n+1$, Bivariate Taylor Interpolation Theorem 8.31 implies that the following $(n+1)$-term point-normalized l'Hôpital dual asymptotic expansion holds:

$$f(x, y) \sim \sum_{i=0}^{n} c_i \cdot g_i(x)\, h_i(y) \quad \text{as} \quad x \to x_0 \quad \text{or} \quad y \to y_0.$$

This establishes the third of the three conclusions of proposition P_{n+1}, assuming the hypotheses of proposition P_{n+1} and the truth of proposition P_n.

We have shown that proposition P_1 holds, and that proposition P_{n+1} follows from proposition P_n for any positive integer n. By induction on n, proposition P_n holds for all positive integers n. ■

Existence Theorem 8.35 asserts that the truth of condition (8.7) for $0 \leq i < n$ is sufficient for point-normalized l'Hôpital dual asymptotic expansion (8.8) to hold. Conversely, Uniqueness Theorem 8.32 asserts that if point-normalized l'Hôpital dual asymptotic expansion (8.8) holds, then condition (8.7) must also hold for $0 \leq i < n$. From this combined analysis, we conclude that $f \in \mathrm{LDAE}^n_{(x_0,y_0)}(X \times Y)$ if and only if

$$(\partial_x^i \partial_y^i r_i)(x_0, y_0) \neq 0 \quad \text{for} \quad 0 \leq i < n,$$

where the sequence of remainders $\{r_i\}_{i=0}^{n} \subset C^{(n,n)}(X \times Y)$ is uniquely determined by the function $f \in C^{(n,n)}(X \times Y)$ and the point $(x_0, y_0) \in X \times Y$ by recursion on condition (8.7) for $0 \leq i < n$. We thus have a simple but definitive criterion for the existence of the unique point-normalized l'Hôpital dual asymptotic expansion of f to n terms at (x_0, y_0).

This criterion for existence yields an iterative *decision algorithm* for determining whether a given function $f \in C^{(n,n)}(X \times Y)$ is a member of the function class $\mathrm{LDAE}^{\,n}_{(x_0,y_0)}(X \times Y)$. If indeed $f \in \mathrm{LDAE}^n_{(x_0,y_0)}(X \times Y)$, then this decision algorithm will also carry out the following four tasks:

1. The algorithm will *generate* the relevant univariate functions $\{g_i\}_{i=0}^{n-1} \subset C^n(X)$ and $\{h_i\}_{i=0}^{n-1} \subset C^n(Y)$.

2. The algorithm will *certify implicitly* that the sequence $\{g_i\}_{i=0}^{n-1}$ is both l'Hôpital and point-normalized at x_0 and that the sequence $\{h_i\}_{i=0}^{n-1}$ is both l'Hôpital and point-normalized at y_0.

3. The algorithm will *calculate* the coefficients $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}\backslash\{0\}$.

4. The algorithm will *certify implicitly* that point-normalized l'Hôpital dual asymptotic expansion (8.8) holds.

We now give the full specification of this decision algorithm, which is an adaptation of Strong Dual Asymptotic Expansion Algorithm 7.33. Uniqueness Theorem 8.32 and Existence Theorem 8.35 prove that this new algorithm is correct:

Algorithm 8.36 (l'Hôpital Dual Asymptotic Expansion) *Let $f \in C^{(n,n)}(X \times Y)$ and assume that $1 \leq n < \infty$. If one of the steps below causes the algorithm to abort with an error, we can conclude that $f \notin \mathrm{LDAE}^n_{(x_0,y_0)}(X \times Y)$. If the algorithm terminates normally, we can conclude that $f \in \mathrm{LDAE}^n_{(x_0,y_0)}(X \times Y)$; in that case, the algorithm will return the unique point-normalized l'Hôpital dual asymptotic expansion of f to n terms at (x_0, y_0).*

1. *Let $s_0 := 0$ and $r_0 := f$.*
2. *Iterate the following six steps for $i := 0, 1, \ldots, n-1$:*
   (a) *Calculate the coefficient $c_i := (\partial_x^i \partial_y^i r_i)(x_0, y_0)$.*
   (b) *If $c_i = 0$, then abort the algorithm with an error.*
   (c) *Calculate the univariate function $g_i := (E^{y_0} \partial_y^i r_i)/c_i$.*
   (d) *Calculate the univariate function $h_i := (E_{x_0} \partial_x^i r_i)/c_i$.*
   (e) *Calculate the next partial sum by $s_{i+1} := s_i + c_i \cdot (g_i \otimes h_i)$.*
   (f) *Calculate the next remainder by $r_{i+1} := f - s_{i+1}$.*
3. *Return the point-normalized l'Hôpital dual asymptotic expansion s_n.*

There are various modifications of the previous algorithm which are convenient in applications. *One such modification exploits symmetry in order to increase efficiency.* Let $Y = X$, and assume that the function $f \in C^{(n,n)}(X^2)$ is symmetric, meaning that $f(x, y) \equiv f(y, x)$. In addition, select an expansion point $(x_0, x_0) \in X^2$ on the axis of symmetry $y = x$. If $f \in \mathrm{LDAE}^n_{(x_0,x_0)}(X^2)$, then the l'Hôpital dual asymptotic expansion of f to n terms at (x_0, x_0) will satisfy $h_i = g_i$ for $0 \leq i < n$, which means that we have only one univariate function to determine at each iteration instead of two. *Another convenient modification is to generate the terms without point-normalization for the sake of simplicity.* The symmetric, unnormalized version of the previous algorithm is as follows:

Algorithm 8.37 (Symmetric l'Hôpital Dual Asymptotic Expansion)
Let $f \in C^{(n,n)}(X^2)$ be a symmetric function, and assume that $1 \leq n < \infty$. If one

of the steps below causes the algorithm to abort with an error, we can conclude that $f \notin \mathrm{LDAE}^n_{(x_0,x_0)}(X^2)$*. If the algorithm terminates normally, we can conclude that* $f \in \mathrm{LDAE}^n_{(x_0,x_0)}(X^2)$*; in that case, the algorithm will return the unique unnormalized l'Hôpital dual asymptotic expansion of* f *to* n *terms at* (x_0, x_0)*.*

1. *Let* $s_0 := 0$ *and* $r_0 := f$*.*
2. *Iterate the following five steps for* $i := 0, 1, \ldots, n-1$*:*
   (a) *Calculate the univariate function* $g_i := E^{x_0} \partial_y^i r_i$*.*
   (b) *Calculate the coefficient* $c_i := D^i g_i(x_0)$*.*
   (c) *If* $c_i = 0$*, then abort the algorithm with an error.*
   (d) *Calculate the next partial sum by* $s_{i+1} := s_i + (g_i \otimes g_i)/c_i$*.*
   (e) *Calculate the next remainder by* $r_{i+1} := f - s_{i+1}$*.*
3. *Return the unnormalized l'Hôpital dual asymptotic expansion* s_n*.*

How does Symmetric l'Hôpital Dual Asymptotic Expansion Algorithm 8.37 actually differ from l'Hôpital Dual Asymptotic Expansion Algorithm 8.36 in the special case $Y = X$ and $y_0 = x_0$? These two algorithms both compute the same coefficient c_i but use different formulas:

$$c_i := D^i g_i(x_0) = (\partial_x^i \partial_y^i r_i)(x_0, x_0).$$

This means that both algorithms use the same criterion for deciding whether the symmetric function f is a member of $\mathrm{LDAE}^n_{(x_0,x_0)}(X^2)$. In addition, the unnormalized terms generated by the symmetric algorithm can easily be transformed into point-normalized form via

$$(g_i \otimes g_i)/c_i = c_i \cdot [(g_i/c_i) \otimes (g_i/c_i)].$$

Although the symmetric, unnormalized algorithm is both more efficient and simpler than the original algorithm, we conclude that the differences between these two algorithms are not so substantial as to require separate theoretical justifications.

With the previous two algorithms at our disposal, we now have simple methods for generating l'Hôpital dual asymptotic expansions whenever they exist. These methods are based on normal differentiation, which is a symbolic operation that we can carry out easily—especially with the help of a computer algebra system such as Maple.

In summary, the iterative algorithms described above give us easy ways to perform Taylor interpolation on two rectilinear coordinate lines using natural tensor products, and thus facilitate the separation of variables in applications. The successful use of these techniques in applications is based on the fact that bivariate Taylor interpolation generally yields a good local approximation of the original function near the rectilinear coordinate lines under consideration. In the next subsection, we will learn how to estimate this approximation error rigorously and will describe its asymptotic behavior.

8.3.3 Remainder and Convergence Theories

In this subsection, we will develop the remainder theory for bivariate Taylor interpolation on two rectilinear coordinate lines by building upon the remainder theory for univariate Taylor interpolation at a point. We will then use the resulting bivariate remainder theory to give a sufficient condition for the uniform convergence of l'Hôpital dual asymptotic expansions with infinitely many terms.

We will now prove the following remainder theorem for bivariate Taylor interpolation by adapting a proof technique found in [Tho76]. The key idea is to parametrically extend univariate integral representations of the remainder in each variable separately, and then combine the two new integral representations by substituting a modified version of one into the other. The derivation of the mean-value form of the remainder for bivariate Taylor interpolation is slightly more involved, but follows a similar pattern.

Theorem 8.38 (Bivariate Taylor Remainder) *Let $X, Y \subset \mathbb{R}$ be intervals, let $x_0 \in X$ and $y_0 \in Y$, and assume that $f_1, f_2 \in C^{(n,n)}(X \times Y)$, where n is a positive integer. If f_1 Taylor interpolates f_2 to order $n-1$ on $x = x_0$ and $y = y_0$, then the remainder $r := f_1 - f_2$ satisfies*

$$r(x,y) = \frac{1}{[(n-1)!]^2} \cdot \int_{x_0}^{x} \int_{y_0}^{y} (x-s)^{n-1} \, (y-t)^{n-1} \, (\partial_x^n \partial_y^n r)(s,t) \, dt \, ds$$

for all $(x,y) \in X \times Y$. In addition, for each point $(x_1, y_1) \in [X \backslash \{x_0\}] \times [Y \backslash \{y_0\}]$, there exists a point (ξ, η) in the open rectangle with vertices $\{(x_i, y_j) : 0 \le i, j \le 1\}$ such that the remainder satisfies

$$r(x_1, y_1) = \frac{(\partial_x^n \partial_y^n r)(\xi, \eta)}{(n!)^2} \cdot (x_1 - x_0)^n (y_1 - y_0)^n.$$

If $\left\|\partial_x^n \partial_y^n r\right\|_\infty < \infty$, *then we obtain the rigorous error estimate*

$$|r(x,y)| \leq \frac{\left\|\partial_x^n \partial_y^n r\right\|_\infty}{(n!)^2} \cdot |x - x_0|^n \, |y - y_0|^n \quad \textit{for all} \quad (x,y) \in X \times Y,$$

which in turn implies the following asymptotic behavior:

$$r(x,y) = O\left((x - x_0)^n (y - y_0)^n\right) \quad \textit{for all} \quad (x,y) \in X \times Y.$$

In addition, if $|x - x_0| \, |y - y_0| \leq \rho$ *for all* $(x,y) \in X \times Y$, *then we obtain the following uniform error estimate:*

$$\|r\|_\infty \leq \frac{\left\|\partial_x^n \partial_y^n r\right\|_\infty}{(n!)^2} \cdot \rho^n.$$

Proof. For each fixed $y \in Y$, the y-section $(f_1)^y \in C^n(X)$ Taylor interpolates the y-section $(f_2)^y \in C^n(X)$ to order $n-1$ at $x_0 \in X$. Univariate Taylor Remainder Theorem 8.10 implies that the remainder $r^y = (f_1)^y - (f_2)^y$ satisfies

$$r^y(x) = \frac{1}{(n-1)!} \cdot \int_{x_0}^{x} (x - s)^{n-1} \, D^n r^y(s) \, ds \quad \text{for all} \quad x \in X.$$

Since the previous equation holds for all $y \in Y$, we can rewrite this as

$$r(x,y) = \frac{1}{(n-1)!} \cdot \int_{x_0}^{x} (x - s)^{n-1} \, \partial_x^n r(s,y) \, ds \quad \text{for all} \quad (x,y) \in X \times Y. \tag{8.9}$$

A similar argument based on x-sections for each fixed $x \in X$ yields

$$r(x,y) = \frac{1}{(n-1)!} \cdot \int_{y_0}^{y} (y - t)^{n-1} \, \partial_y^n r(x,t) \, dt \quad \text{for all} \quad (x,y) \in X \times Y.$$

Since the integrand in the previous equation depends on the parameter x in a continuously differentiable fashion to order n, Leibniz's rule for the differentiation of a definite integral with respect to a parameter assures us that the n-th order partial derivative operator ∂_x^n commutes with the definite integral operator $\int_{y_0}^{y} \bullet \, dt$. Applying the operator ∂_x^n to both sides of the previous equation and substituting $x = s$ yields

$$\partial_x^n r(s,y) = \frac{1}{(n-1)!} \cdot \int_{y_0}^{y} (y - t)^{n-1} \, (\partial_x^n \partial_y^n r)(s,t) \, dt \quad \text{for all} \quad (s,y) \in X \times Y. \tag{8.10}$$

Substituting equation (8.10) into equation (8.9) yields

$$\begin{aligned} r(x,y) &= \frac{1}{(n-1)!} \cdot \int_{x_0}^{x} (x-s)^{n-1} \frac{1}{(n-1)!} \cdot \int_{y_0}^{y} (y-t)^{n-1} \, (\partial_x^n \partial_y^n r)(s,t) \, dt \, ds \\ &= \frac{1}{[(n-1)!]^2} \cdot \int_{x_0}^{x} \int_{y_0}^{y} (x-s)^{n-1} \, (y-t)^{n-1} \, (\partial_x^n \partial_y^n r)(s,t) \, dt \, ds, \end{aligned}$$

which holds for all $(x,y) \in X \times Y$, as desired.

Now fix a point $(x_1, y_1) \in [X \backslash \{x_0\}] \times [Y \backslash \{y_0\}]$. Since the y_1-section $(f_1)^{y_1} \in C^n(X)$ Taylor interpolates the y_1-section $(f_2)^{y_1} \in C^n(X)$ to order $n-1$ at $x_0 \in X$, Univariate Taylor Remainder Theorem 8.10 implies that there exists some $\xi \in X$ such that the remainder $r^{y_1} = (f_1)^{y_1} - (f_1)^{y_2}$ satisfies

$$r^{y_1}(x_1) = \frac{D^n r^{y_1}(\xi)}{n!} \cdot (x_1 - x_0)^n \quad \text{where} \quad \min x_i < \xi < \max x_i.$$

Note that we can rewrite the previous equation in terms of the n-th order normal derivative of r on the line $x = \xi$ as follows:

$$r(x_1, y_1) = \frac{\partial_x^n r(\xi, y_1)}{n!} \cdot (x_1 - x_0)^n = \frac{(E_\xi \partial_x^n r)(y_1)}{n!} \cdot (x_1 - x_0)^n. \tag{8.11}$$

Next, we will consider what we know about the normal derivative $E_\xi \partial_x^n r \in C^n(Y)$. We know that $y = y_0$ is a zero line of r of multiplicity at least n, which means that the following n univariate identities hold:

$$\partial_y^i r(x, y_0) \equiv 0 \quad \text{for} \quad 0 \le i < n.$$

If we apply the operator ∂_x^n to both sides of these identities and commute the operators ∂_x^n and ∂_y^i, we obtain the n new identities

$$\partial_y^i (\partial_x^n r)(x, y_0) \equiv 0 \quad \text{for} \quad 0 \le i < n.$$

Substituting $x = \xi$ into these identities yields the n equations

$$\partial_y^i (\partial_x^n r)(\xi, y_0) = D^i (E_\xi \partial_x^n r)(y_0) = 0 \quad \text{for} \quad 0 \le i < n,$$

which tell us that y_0 is a zero of the normal derivative

$$E_\xi \partial_x^n r := E_\xi \partial_x^n f_1 - E_\xi \partial_x^n f_2$$

of multiplicity at least n.

From the previous paragraph, we conclude that the normal derivative $E_\xi \partial_x^n f_1 \in C^n(Y)$ Taylor interpolates the normal derivative $E_\xi \partial_x^n f_2 \in C^n(Y)$ to order $n-1$ at $y_0 \in Y$. Since $y_1 \in Y \backslash \{y_0\}$, Univariate Taylor Remainder Theorem 8.10 implies that there exists some $\eta \in Y$ such that the remainder $E_\xi \partial_x^n r$ satisfies

$$(E_\xi \partial_x^n r)(y_1) = \frac{D^n(E_\xi \partial_x^n r)(\eta)}{n!} \cdot (y_1 - y_0)^n \quad \text{where} \quad \min y_i < \eta < \max y_i.$$

We can rewrite the previous equation as

$$(E_\xi \partial_x^n r)(y_1) = \frac{\partial_y^n(\partial_x^n r)(\xi,\eta)}{n!} \cdot (y_1 - y_0)^n = \frac{(\partial_x^n \partial_y^n r)(\xi,\eta)}{n!} \cdot (y_1 - y_0)^n. \tag{8.12}$$

Substituting equation (8.12) into equation (8.11) yields

$$\begin{aligned} r(x_1, y_1) &= \frac{\dfrac{(\partial_x^n \partial_y^n r)(\xi,\eta)}{n!} \cdot (y_1 - y_0)^n}{n!} \cdot (x_1 - x_0)^n \\ &= \frac{(\partial_x^n \partial_y^n r)(\xi,\eta)}{(n!)^2} \cdot (x_1 - x_0)^n (y_1 - y_0)^n, \end{aligned}$$

where the point (ξ, η) lies in the open rectangle with vertices $\{(x_i, y_j) : 0 \le i, j \le 1\}$, as desired. The remaining conclusions of the theorem follow easily from the mean-value form of the remainder. ■

We will now use the previous theorem to develop the following sufficient condition for uniform convergence on compact rectangles for l'Hôpital dual asymptotic expansions with infinitely many terms:

Theorem 8.39 (Bivariate Taylor Convergence) *Let $[a, b], [c, d] \subset \mathbb{R}$ be compact intervals, let $R := [a, b] \times [c, d]$ be a compact rectangle, and let $(x_0, y_0) \in R$. Assume that $f \in C^\infty(R)$ has the following l'Hôpital dual asymptotic expansion*

$$f(x,y) \sim \sum_{i=0}^{\infty} c_i \cdot g_i(x)\, h_i(y) \quad \text{as} \quad x \to x_0 \quad \text{or} \quad y \to y_0,$$

and let $\{r_n\}_{n=0}^{\infty} \subset C^{\infty}(R)$ *denote the sequence of remainders. If the following asymptotic order relation holds*

$$\left\|\partial_x^n \partial_y^n r_n\right\|_{\infty} = O(n!) \quad as \quad n \to \infty,$$

then the formal infinite series

$$s_{\infty} := \sum_{i=0}^{\infty} c_i \cdot (g_i \otimes h_i)$$

converges uniformly on R, *and the sum of the infinite series* s_{∞} *equals the function* f.

Proof. We can assume without loss of generality that the given l'Hôpital dual asymptotic expansion is point-normalized at (x_0, y_0). Bivariate Taylor Interpolation Theorem 8.31 implies that the point-normalized l'Hôpital dual asymptotic expansion s_{∞} Taylor interpolates f to order ∞ on $x = x_0$ and $y = y_0$, which by definition means that for all $n \geq 1$, the partial sum s_n Taylor interpolates f to order $n-1$ on $x = x_0$ and $y = y_0$. Since $r_n \in C^{\infty}(R)$ and R is compact, we know that $\left\|\partial_x^n \partial_y^n r_n\right\|_{\infty} < \infty$. Bivariate Taylor Remainder Theorem 8.38 with $\rho := (b-a)(d-c)$ therefore yields the uniform error estimate

$$\|r_n\|_{\infty} \leq \frac{\left\|\partial_x^n \partial_y^n r_n\right\|_{\infty}}{(n!)^2} \cdot [(b-a)(d-c)]^n \quad \text{for all} \quad n \geq 0.$$

Since $\left\|\partial_x^n \partial_y^n r_n\right\|_{\infty} = O(n!)$ as $n \to \infty$, there exists a real constant $M > 0$ and a natural number N such that

$$\left\|\partial_x^n \partial_y^n r_n\right\|_{\infty} \leq M \cdot n! \quad \text{for all} \quad n \geq N.$$

The previous two inequalities together imply that

$$\|r_n\|_{\infty} \leq M \cdot \frac{[(b-a)(d-c)]^n}{n!} \quad \text{for all} \quad n \geq N.$$

This new inequality and the limit

$$\lim_{n \to \infty} \frac{[(b-a)(d-c)]^n}{n!} = 0$$

together imply that $\|r_n\|_{\infty} \to 0$ as $n \to \infty$. We conclude that $s_n \to f$ uniformly on R as $n \to \infty$. ■

The previous theorem tells us that even if the sequence $\{\left\|\partial_x^n \partial_y^n r_n\right\|_{\infty}\}_{n=0}^{\infty}$ grows at a factorial rate, the formal infinite series s_{∞} is guaranteed to converge uniformly to f on the compact rectangle R. Of course, if the sequence $\{\left\|\partial_x^n \partial_y^n r_n\right\|_\infty\}_{n=0}^\infty$ is bounded, or grows at a logarithmic, polynomial, or exponential rate, we can also conclude that the infinite series s_∞ converges uniformly to f on R. In the next subsection, we will use this general result to establish the uniform convergence of a well-known addition formula for the Bessel function J_0.

8.3.4 An Application to Bessel Functions of the First Kind

Let J_n denote the Bessel function of the first kind of order n for any $n \in \mathbb{Z}$. The following infinite series expansion is a special case of Equation 9.1.75 in [AS64, p. 363, Dover]:

$$J_0(x+y) = \sum_{i=-\infty}^{\infty} J_{-i}(x)\, J_i(y) \quad \text{for all} \quad (x,y) \in \mathbb{R}^2.$$

This expansion is referred to as **Neumann's addition theorem for** J_0 in [AS64, p. 363, Dover]. Using the transformation $J_{-i} = (-1)^i \cdot J_i$, we can rewrite Neumann's addition theorem as

$$J_0(x+y) = J_0(x)\, J_0(y) + 2\sum_{i=1}^{\infty}(-1)^i \cdot J_i(x)\, J_i(y) \quad \text{for all} \quad (x,y) \in \mathbb{R}^2. \tag{8.13}$$

The second expansion is referred to as the **addition formula for** J_0 in [Bow58, p. 90], and both of these expansions are referred to collectively as **addition theorems for** J_0 in [Arf85, p. 585]. These infinite series expansions converge pointwise on the entire plane, and converge uniformly on every bounded subset of the plane. In this subsection, we will give an original derivation and proof of addition formula (8.13) using the theory of l'Hôpital dual asymptotic expansions.

The author originally proposed this method for deriving and proving the addition formula for J_0 in his master's thesis [Cha98, pp. 125–128], where he noted that expansion (8.13) is actually the dual asymptotic expansion of the function $f(x,y) := J_0(x+y)$ to infinitely many terms at the point $(0,0)$. There were two obstacles that prevented this new method from being carried out successfully at that time:

1. Generating the terms of the dual asymptotic expansion directly from the definition of the asymptotic splitting operator $\Upsilon_{(0,0)}$ proved to be unwieldy after the first two terms; generating the general n-th term of the expansion would require a much simpler method.

2. In order to prove that the infinite series expansion converged uniformly to f on every bounded subset of the plane, some kind of convergence theory was needed; this did not exist at the time of the original proposal.

This doctoral thesis provides the means to overcome both of these obstacles: Symmetric l'Hôpital Dual Asymptotic Expansion Algorithm 8.37 gives us the means to generate the terms of the expansion from the normal derivatives $\{E^0 \partial_y^n r_n\}_{n=0}^{\infty}$, and Bivariate Taylor Convergence Theorem 8.39 gives us the means to prove the uniform convergence of the expansion by determining the rate of growth of the numerical sequence $\{\left\|\partial_x^n \partial_y^n r_n\right\|_\infty\}_{n=0}^{\infty}$.

We will first derive the right-hand side of series expansion (8.13) from the left-hand side using Symmetric l'Hôpital Dual Asymptotic Expansion Algorithm 8.37. This is a true *derivation* because the symmetric algorithm actually *generates* the terms on the right-hand side using only the given closed-form expression $J_0(x+y)$ on the left-hand side and the expansion point $(0,0)$.

In order to apply the symmetric algorithm, we need to know only three fundamental facts about the Bessel function J_n: the transformation $J_{-n} = (-1)^n \cdot J_n$, the value of $J_n(0)$, and a formula for the higher-order derivatives $D^m J_n$. The desired value is given by

$$J_n(0) = \delta_{n0} := \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases} \quad \text{for all} \quad n \in \mathbb{Z}.$$

For the first derivative, we refer to Equation 9.1.27 in [AS64, p. 361, Dover], which states that

$$DJ_n = \frac{1}{2}(J_{n-1} - J_{n+1}) \quad \text{for all} \quad n \in \mathbb{Z}. \tag{8.14}$$

When $n = 0$, the previous equation and the transformation $J_{-1} = -J_1$ together yield Equation 9.1.28 in [AS64, p. 361, Dover], which states that

$$DJ_0 = -J_1. \tag{8.15}$$

Repeated application of equation (8.14) yields Equation 9.1.31 in [AS64, p. 361, Dover], which states that

$$D^m J_n = \frac{1}{2^m} \sum_{i=0}^{m} (-1)^i \binom{m}{i} \cdot J_{n-m+2i} \quad \text{for all} \quad m \in \mathbb{N},\ n \in \mathbb{Z}. \tag{8.16}$$

Evaluating both sides of the previous equation at zero yields the following very useful result, which is valid for all $m \in \mathbb{N}$ and $n \in \mathbb{Z}$:

$$D^m J_n(0) = \begin{cases} \dfrac{(-1)^{(m-n)/2}}{2^m} \dbinom{m}{(m-n)/2} & \text{if} \quad m \equiv n \bmod 2 \quad \text{and} \quad m \geq |n| \\ \\ 0 & \text{if} \quad m \not\equiv n \bmod 2 \quad \text{or} \quad m < |n| \,. \end{cases} \tag{8.17}$$

We will find the following lemma of considerable help in our derivation of series expansion (8.13):

Lemma 8.40 *For each $n \in \mathbb{N}$, the derivative $D^n J_0$ can be written as a linear combination of the Bessel functions $\{J_i\}_{i=0}^n$ via*

$$D^n J_0 = D^n J_0(0) \cdot J_0 + (-1)^n \cdot 2 \sum_{i=1}^{n} D^n J_i(0) \cdot J_i. \tag{8.18}$$

Proof. We will prove the result by mathematical induction on n. For each $n \in \mathbb{N}$, let P_n denote the proposition that equation (8.18) holds. When $n = 0$, equation (8.18) reduces to $J_0 = 1 \cdot J_0 + 0$; since this is obviously true, proposition P_0 holds.

Suppose that proposition P_n holds for an arbitrary $n \in \mathbb{N}$. We must show that proposition P_{n+1} follows. If we apply the operator D to equation (8.18) and use equations (8.15) and (8.14) to simplify the result, we obtain

$$\begin{aligned} D^{n+1} J_0 &= D(D^n J_0) = D^n J_0(0) \cdot DJ_0 + (-1)^n \cdot 2 \sum_{i=1}^{n} D^n J_i(0) \cdot DJ_i \\ &= -D^n J_0(0) \cdot J_1 + (-1)^n \cdot \sum_{i=1}^{n} D^n J_i(0) \cdot (J_{i-1} - J_{i+1}) \\ &= (-1)^n \cdot \sum_{i=1}^{n} D^n J_i(0) \cdot J_{i-1} + \left(-D^n J_0(0) \cdot J_1 + (-1)^{n+1} \cdot \sum_{i=1}^{n} D^n J_i(0) \cdot J_{i+1} \right). \end{aligned}$$

If n is odd, equation (8.17) implies that $D^n J_0(0) = 0$. If n is even, then $(-1)^{n+1} = -1$. These two cases together imply that

$$-D^n J_0(0) = (-1)^{n+1} D^n J_0(0) \quad \text{for all} \quad n \in \mathbb{N}.$$

Using this result to continue our analysis, we obtain the following by reindexing the two summations and regrouping the terms:

$$\begin{aligned} D^{n+1}J_0 &= \left(-(-1)^{n+1}\cdot\sum_{j=0}^{n-1}D^nJ_{j+1}(0)\cdot J_j\right) \\ &\quad+\left((-1)^{n+1}D^nJ_0(0)\cdot J_1+(-1)^{n+1}\cdot\sum_{j=2}^{n+1}D^nJ_{j-1}(0)\cdot J_j\right) \\ &= \left(-(-1)^{n+1}\,D^nJ_1(0)\cdot J_0-(-1)^{n+1}\cdot\sum_{j=1}^{n-1}D^nJ_{j+1}(0)\cdot J_j\right) \\ &\quad+\left((-1)^{n+1}\cdot\sum_{j=1}^{n+1}D^nJ_{j-1}(0)\cdot J_j\right). \end{aligned}$$

Equation (8.17) also implies that

$$-(-1)^{n+1}D^nJ_1(0)=-D^nJ_1(0)\quad\text{for all}\quad n\in\mathbb{N}.$$

Using this result to continue our analysis, we obtain the following by collecting the coefficients of the functions $\{J_j\}_{j=1}^{n-1}$ into a single summation:

$$\begin{aligned} D^{n+1}J_0 &= -D^nJ_1(0)\cdot J_0+(-1)^{n+1}\cdot\sum_{j=1}^{n-1}D^n(J_{j-1}-J_{j+1})(0)\cdot J_j \\ &\quad+(-1)^{n+1}\cdot\left(D^nJ_{n-1}(0)\cdot J_n+D^nJ_n(0)\cdot J_{n+1}\right). \end{aligned}$$

We can now use equations (8.15) and (8.14) in reverse to transform the previous result into

$$\begin{aligned} D^{n+1}J_0 &= D^{n+1}J_0(0)\cdot J_0+(-1)^{n+1}\cdot 2\sum_{j=1}^{n-1}D^{n+1}J_j(0)\cdot J_j \\ &\quad+(-1)^{n+1}\cdot\left(D^nJ_{n-1}(0)\cdot J_n+D^nJ_n(0)\cdot J_{n+1}\right). \end{aligned}$$

Equation (8.17), which is an inexhaustible fountain of life-saving trivialities, also implies that

$$D^nJ_{n-1}(0)=0=2D^{n+1}J_n(0)\quad\text{and}\quad D^nJ_n(0)=2^{-n}=2D^{n+1}J_{n+1}(0)\quad\text{for all}\quad n\in\mathbb{N}.$$

We can use these results to complete our analysis as follows:

$$\begin{aligned} D^{n+1}J_0 &= D^{n+1}J_0(0)\cdot J_0 + (-1)^{n+1}\cdot 2\sum_{j=1}^{n-1} D^{n+1}J_j(0)\cdot J_j \\ &\quad +(-1)^{n+1}\cdot\left(2D^{n+1}J_n(0)\cdot J_n + 2D^{n+1}J_{n+1}(0)\cdot J_{n+1}\right) \\ &= D^{n+1}J_0(0)\cdot J_0 + (-1)^{n+1}\cdot 2\sum_{j=1}^{n+1} D^{n+1}J_j(0)\cdot J_j. \end{aligned}$$

This last result asserts that equation (8.18) holds with n replaced by $n+1$, which proves proposition P_{n+1}.

We have shown that proposition P_0 holds, and we have shown that proposition P_{n+1} follows from proposition P_n for each $n \in \mathbb{N}$. By mathematical induction on n, we conclude that proposition P_n holds for all $n \in \mathbb{N}$. ■

The previous lemma can also be proven in a completely different way by letting $n = 0$ in equation (8.16) and considering two independent cases corresponding to whether m is odd or even. This approach uses the transformations $J_{2i-m} = (-1)^{m-2i}\cdot J_{m-2i}$ and $\binom{m}{i} = \binom{m}{m-i}$ to collect duplicate terms in equation (8.16) into a new summation with essentially half as many terms, all of which are linearly independent. After reindexing the new summation, equation (8.17) can be used in reverse to express the coefficients in terms of the constants $\{D^m J_i(0)\}_{i=0}^m$. This alternate proof is perhaps more straightforward than the induction proof given above, but it is also more tedious, and is therefore omitted.[1]

With all these handy facts about Bessel functions at our disposal, we can now carry out the following derivation of series expansion (8.13):

Theorem 8.41 *If we define $f \in C^\omega(\mathbb{R}^2)$ by $f(x,y) := J_0(x+y)$ for all $(x,y)\in\mathbb{R}^2$, then $f \in \mathrm{LDAE}^\infty_{(0,0)}(\mathbb{R}^2)$, and the l'Hôpital dual asymptotic expansion of f to infinitely many terms at $(0,0)$ is*

$$J_0(x+y) \sim J_0(x)\,J_0(y) + 2\sum_{i=1}^{\infty}(-1)^i\,J_i(x)\,J_i(y) \quad \text{as} \quad x\to 0 \quad \text{or} \quad y \to 0.$$

Proof. For each positive integer n, let P_n denote the logical proposition that $f \in \mathrm{LDAE}^n_{(0,0)}(\mathbb{R}^2)$ and that the l'Hôpital dual asymptotic expansion of f to n terms

[1] The author finds the alternate proof more tedious because it involves more frequent and more complicated reindexing of the resulting summations. In contrast, the induction proof has many more unexpected twists and turns, and is thus a much better entertainment value!

at $(0,0)$ is

$$J_0(x+y) \sim J_0(x)\, J_0(y) + 2\sum_{i=1}^{n-1}(-1)^i\, J_i(x)\, J_i(y) \quad \text{as} \quad x \to 0 \quad \text{or} \quad y \to 0.$$

It suffices to prove that proposition P_n holds for all positive integers n. We will show this by mathematical induction on n using Symmetric l'Hôpital Dual Asymptotic Expansion Algorithm 8.37.

We must first show that proposition P_1 holds. Let us initialize the symmetric algorithm by defining $s_0 := 0$ and $r_0 := f$. When $i = 0$, we obtain

$$g_0 := E^0 \partial_y^0 r_0 = E^0 f = J_0 \quad \text{and} \quad c_0 := D^0 g_0(0) = J_0(0) = 1.$$

Since $c_0 \neq 0$, it follows that $f \in \mathrm{LDAE}^1_{(0,0)}(\mathbb{R}^2)$, and we obtain

$$s_1 := s_0 + (g_0 \otimes g_0)/c_0 = J_0 \otimes J_0.$$

This proves that proposition P_1 holds.

Suppose that proposition P_n holds for an arbitrary positive integer n. We must show that proposition P_{n+1} follows. Proposition P_n implies that after n iterations of the symmetric algorithm, we obtain

$$s_n := J_0 \otimes J_0 + 2\sum_{i=1}^{n-1}(-1)^i \cdot (J_i \otimes J_i). \tag{8.19}$$

When $i = n$ in iteration $n+1$, the symmetric algorithm yields

$$g_n := E^0 \partial_y^n r_n = E^0 \partial_y^n f - E^0 \partial_y^n s_n = D^n J_0 - E^0 \partial_y^n s_n. \tag{8.20}$$

We will now consider how to calculate the terms $D^n J_0$ and $E^0 \partial_y^n s_n$ separately.

Lemma 8.40 states that

$$D^n J_0 = D^n J_0(0) \cdot J_0 + (-1)^n \cdot 2\sum_{i=1}^{n} D^n J_i(0) \cdot J_i. \tag{8.21}$$

Applying the normal derivative operator $E^0 \partial_y^n$ to equation (8.19) yields

$$E^0 \partial_y^n s_n = D^n J_0(0) \cdot J_0 + 2 \sum_{i=1}^{n-1} (-1)^i \, D^n J_i(0) \cdot J_i. \tag{8.22}$$

If $n \not\equiv i \bmod 2$, equation (8.17) implies that $D^n J_i(0) = 0$. If $n \equiv i \bmod 2$, it follows that $(-1)^n = (-1)^i$. These two cases together imply that

$$(-1)^i \, D^n J_i(0) = (-1)^n \, D^n J_i(0) \quad \text{for all} \quad i, n \in \mathbb{N}.$$

We can this result to rewrite equation (8.22) as

$$E^0 \partial_y^n s_n = D^n J_0(0) \cdot J_0 + (-1)^n \cdot 2 \sum_{i=1}^{n-1} D^n J_i(0) \cdot J_i. \tag{8.23}$$

Subtracting equation (8.23) from equation (8.21) and cancelling repeated terms yields the following result by equation (8.20):

$$g_n = D^n J_0 - E^0 \partial_y^n s_n = (-1)^n \cdot 2 \cdot D^n J_n(0) \cdot J_n.$$

We can now calculate the coefficient via

$$c_n := D^n g_n(0) = (-1)^n \cdot 2 \cdot [D^n J_n(0)]^2.$$

Since $D^n J_n(0) = 2^{-n}$ by equation (8.17), it follows that $c_n \neq 0$, and we conclude that $f \in \mathrm{LDAE}^{n+1}_{(0,0)}(\mathbb{R}^2)$. The next term in the expansion is given by

$$\frac{1}{c_n} \cdot (g_n \otimes g_n) = \frac{(-1)^{2n} \cdot 4 \cdot [D^n J_n(0)]^2}{(-1)^n \cdot 2 \cdot [D^n J_n(0)]^2} \cdot (J_n \otimes J_n) = (-1)^n \cdot 2 \cdot (J_n \otimes J_n),$$

and the next partial sum is given by

$$\begin{aligned} s_{n+1} &:= s_n + \frac{1}{c_n} \cdot (g_n \otimes g_n) \\ &= J_0 \otimes J_0 + \left(2 \sum_{i=1}^{n-1} (-1)^i \cdot (J_i \otimes J_i) \right) + (-1)^n \cdot 2 \cdot (J_n \otimes J_n) \\ &= J_0 \otimes J_0 + 2 \sum_{i=1}^{n} (-1)^i \cdot (J_i \otimes J_i). \end{aligned}$$

This proves that proposition P_{n+1} holds.

We have shown that proposition P_1 holds, and we have shown that proposition P_{n+1} follows from proposition P_n for each positive integer n. By mathematical induction on n, we conclude that proposition P_n holds for all positive integers n. ■

A direct consequence of the previous theorem is that the sequence $\{J_n\}_{n=0}^{\infty}$ is l'Hôpital at 0, which means that 0 is a zero of J_n of multiplicity n for all $n \in \mathbb{N}$. This also follows directly from equation (8.17), our one-stop shop for fascinating Bessel function facts.[2]

In order to prove the convergence properties of series expansion (8.13), we need bounds on the higher-order derivatives $D^m J_n$. We can easily obtain such bounds from the following integral representation, which appears as Equation 9.1.21 in [AS64, pp. 360, Dover]:

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin t - nt)\, dt \quad \text{for all} \quad n \in \mathbb{N},\ x \in \mathbb{R}.$$

Applying the operator D^m to the previous equation yields a new integral representation which is valid for all $m, n \in \mathbb{N}$ and $x \in \mathbb{R}$:

$$D^m J_n(x) = \frac{1}{\pi} \int_0^{\pi} \partial_x^m \cos(x \sin t - nt)\, dt = \frac{1}{\pi} \int_0^{\pi} \pm \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (x \sin t - nt) \cdot \sin^m t\, dt.$$

From this, we obtain the following estimate, which is valid for all $m, n \in \mathbb{N}$:

$$|D^m J_n(x)| \le \frac{1}{\pi} \int_0^{\pi} \left| \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (x \sin t - nt) \cdot \sin^m t \right| dt \le \frac{1}{\pi} \int_0^{\pi} dt = 1 \quad \text{for all} \quad x \in \mathbb{R}.$$

[2] Try saying that out loud ten times quickly! How many times can you say it before you start sounding like Jimmy Stewart?

Taking the supremum of this estimate over the entire real line yields the final estimate

$$\|D^m J_n\|_\infty \leq 1 \quad \text{for all} \quad m, n \in \mathbb{N}. \tag{8.24}$$

With this bound at our disposal, we are now ready to prove the convergence properties of series expansion (8.13):

Theorem 8.42 *Define $f \in C^\omega(\mathbb{R}^2) \cap \mathrm{LDAE}^\infty_{(0,0)}(\mathbb{R}^2)$ by $f(x,y) := J_0(x+y)$ for all $(x,y) \in \mathbb{R}^2$, and define the compact square $S_\rho := [-\rho, \rho]^2$ for all real $\rho > 0$. The l'Hôpital dual asymptotic expansion*

$$J_0(x+y) \sim J_0(x)\, J_0(y) + 2\sum_{i=1}^{\infty}(-1)^i\, J_i(x)\, J_i(y) \quad \textit{as} \quad x \to 0 \quad \textit{or} \quad y \to 0$$

converges pointwise to f on all of $\mathbb{R}^2$, and converges uniformly to f on every bounded subset of $\mathbb{R}^2$. Furthermore, for all real $\rho > 0$, the restrictions of the remainders $\{r_n\}_{n=1}^\infty \subset C^\omega(\mathbb{R}^2)$ to the compact square S_ρ satisfy

$$\|r_n|S_\rho\|_\infty \leq \frac{2\rho^{2n}}{n!\,(n-1)!} \quad \textit{for all} \quad n \geq 1.$$

Proof. The key to the proof is to estimate the expression $\left\|\partial_x^n \partial_y^n r_n\right\|_\infty$ for all $n \geq 1$. Since $r_n := f - s_n$, we obtain

$$\left\|\partial_x^n \partial_y^n r_n\right\|_\infty \leq \left\|\partial_x^n \partial_y^n f\right\|_\infty + \left\|\partial_x^n \partial_y^n s_n\right\|_\infty.$$

Inequality (8.24) implies that

$$\left\|\partial_x^n \partial_y^n f\right\|_\infty = \left\|D^{2n} J_0\right\|_\infty \leq 1.$$

Applying the operator $\partial_x^n \partial_y^n$ to the partial sum

$$s_n := J_0 \otimes J_0 + 2\sum_{i=1}^{n-1}(-1)^i \cdot (J_i \otimes J_i)$$

yields

$$\partial_x^n \partial_y^n s_n = D^n J_0 \otimes D^n J_0 + 2\sum_{i=1}^{n-1}(-1)^i \cdot (D^n J_i \otimes D^n J_i).$$

This equation in turn yields the following estimate by inequality (8.24):

$$\begin{aligned}\left\|\partial_x^n \partial_y^n s_n\right\|_\infty &\leq \|D^n J_0 \otimes D^n J_0\|_\infty + 2\sum_{i=1}^{n-1} \|D^n J_i \otimes D^n J_i\|_\infty \\ &= \|D^n J_0\|_\infty^2 + 2\sum_{i=1}^{n-1} \|D^n J_i\|_\infty^2 \leq 1 + 2(n-1) = 2n - 1.\end{aligned}$$

From the above analysis, we conclude that

$$\left\|\partial_x^n \partial_y^n r_n\right\|_\infty \leq 1 + (2n-1) = 2n \quad \text{for all} \quad n \geq 1. \tag{8.25}$$

This upper bound implies that

$$\left\|\partial_x^n \partial_y^n r_n\right\|_\infty = O(n) = O(O(n!)) = O(n!) \quad \text{as} \quad n \to \infty.$$

If we apply Bivariate Taylor Convergence Theorem 8.39 to the restrictions of f and s_n to the compact square S_ρ, we conclude that $s_n \to f$ uniformly on S_ρ as $n \to \infty$. If the arbitrary set $S \subset \mathbb{R}^2$ is bounded, then $S \subset S_\rho$ for $\rho > 0$ sufficiently large, which implies that $s_n \to f$ uniformly on S as $n \to \infty$. Similarly, since any point $(x, y) \in \mathbb{R}^2$ satisfies $(x, y) \in S_\rho$ for $\rho > 0$ sufficiently large, and since uniform convergence implies pointwise convergence, it follows that $s_n \to f$ pointwise on $\mathbb{R}^2$ as $n \to \infty$. Finally, given any $\rho > 0$, we note that $|x|\,|y| \leq \rho^2$ for all $(x, y) \in S_\rho$. We can therefore apply Bivariate Taylor Remainder Theorem 8.38 along with inequality (8.25) to obtain the following error estimate:

$$\|r_n|S_\rho\|_\infty \leq \frac{\left\|\partial_x^n \partial_y^n r_n|S_\rho\right\|_\infty}{(n!)^2} \cdot \rho^{2n} \leq \frac{2n}{(n!)^2} \cdot \rho^{2n} = \frac{2\rho^{2n}}{n!\,(n-1)!} \quad \text{for all} \quad n \geq 1.$$

This completes the proof. ■

The previous theorem does more than establish the convergence properties of series expansion (8.13): If we truncate the expansion to n terms, the theorem also gives an explicit and rigorous estimate of the uniform error that results from approximating the function f by the partial sum s_n on the compact square S_ρ. Such error estimates are of great practical value in applications. In the next chapter, we will develop an application which actually attains the smallest error possible—zero!

Chapter 9

Algorithms for Deriving and Proving Identities

The automatic derivation and proof of identities for differentiable functions of several variables has been well studied by researchers in computer algebra. The foundational work of Doron Zeilberger [Zei90] and the more recent work of Frédéric Chyzak and Bruno Salvy [CS98] are particularly notable. The powerful algorithms developed by these researchers are based on the deep and sophisticated theories of holonomic functions and noncommutative Ore algebras—theories which lie well outside the scope of this thesis.

This chapter develops three original algorithms for the automatic derivation and proof of tensor product identities, thereby introducing a new, complementary approach to research in this general area of computer algebra. Since this new approach is much less ambitious than existing approaches, it can therefore use much more elementary methods—this in turn yields a much more accessible, totally self-contained treatment of one particular facet of this interesting subject.

Each of the three algorithms of this chapter uses the same two-phase design: a *derivation phase* followed by a *proof phase*. The derivation phase uses the theories of l'Hôpital, strong, and weak dual asymptotic expansions to generate a tensor product expression from a given closed-form expression in two variables. The proof phase uses an original uniqueness result for the homogeneous hyperbolic eigenproblem to show that the closed-form expression and the tensor product expression are identically equal. Since we have already developed the underlying theory of the derivation phase in the previous two chapters, this chapter begins with a concise but thorough study of the underlying theory of the proof phase.

9.1 The Homogeneous Hyperbolic Eigenproblem

This section begins with the basic definitions that we will need for our study of the automatic derivation and proof of tensor product identities. We will then motivate these derivation and proof techniques with an extended example based on the exponential function. This section concludes with the proof of an original uniqueness result for the homogeneous hyperbolic eigenproblem, which is an initial value problem involving a hyperbolic linear partial differential operator. This uniqueness result will become the cornerstone of the proof phase in all three of the derivation and proof algorithms that we will develop in this chapter.

Let $X, Y \subset \mathbb{R}$ be intervals. Assume that $f \in \mathbb{R}^{X\times Y}$, that $\{g_i\}_{i=0}^{n-1} \subset \mathbb{R}^X$, and that $\{h_i\}_{i=0}^{n-1} \subset \mathbb{R}^Y$, where n is a positive integer. The author proposes that we call an identity of the form

$$f(x,y) = \sum_{i=0}^{n-1} g_i(x)\, h_i(y) \quad \text{for all} \quad (x,y) \in X \times Y$$

a **tensor product identity**. We note in particular that the left-hand side is a closed-form expression in two variables, the right-hand side is a tensor product, and the region of validity is a rectangular subset of the plane. We define the **rank of the tensor product identity** to be the rank of the tensor product on the right-hand side of the identity. If the left-hand side has the special form $f(x,y) := u(x+y)$ for some univariate function u, we call the tensor product identity an **addition formula for** u. Similarly, if the left-hand side has the special form $f(x,y) := u(xy)$, we call the tensor product identity a **multiplication formula for** u. We say that the tensor product identity **is symmetric** if $Y = X$ and the left- and right-hand sides of the identity are both symmetric expressions in the independent variables.[1]

For example, the familiar addition formula for the exponential function is a symmetric tensor product identity of rank one:

$$\exp(x+y) = \exp x \, \exp y \quad \text{for all} \quad (x,y) \in \mathbb{R}^2. \tag{9.1}$$

This identity is also a l'Hôpital dual asymptotic expansion of the function $f(x,y) :=$

[1]Of the more than two dozen distinct tensor product identities which the author has encountered in applications, it is noteworthy that *all* of these could be reduced to symmetric form by a simple linear change of the independent variables.

$\exp(x+y)$ to one term at the point $(0,0)$:

$$\exp(x+y) \sim \exp x \, \exp y \quad \text{as} \quad x \to 0 \quad \text{or} \quad y \to 0. \tag{9.2}$$

Because of the symmetry in this expansion, we can use Symmetric l'Hôpital Dual Asymptotic Expansion Algorithm 8.37 to actually *derive* the right-hand side of identity (9.1) from the left-hand side; applying the symmetric algorithm to the function f and the point $(0,0)$ yields the following after one iteration:

$$\begin{aligned} g_0(x) &:= (E^0 \partial_y^0 r_0)(x) = (E^0 f)(x) = \exp(x+0) = \exp x \\ c_0 &:= D^0 g_0(0) = \exp 0 = 1 \neq 0 \\ s_1(x,y) &:= s_0(x,y) + [g_0(x)\, g_0(y)]/c_0 = \exp x \, \exp y. \end{aligned}$$

We can *prove* that expansion (9.2) is a tensor product identity by showing that the remainder

$$r_1(x,y) := \exp(x+y) - \exp x \, \exp y$$

vanishes on $\mathbb{R}^2$. We begin the proof by noting that Symmetric l'Hôpital Dual Asymptotic Expansion Algorithm 8.37 guarantees that the partial sum s_1 is a l'Hôpital dual asymptotic expansion of f to one term at $(0,0)$. Since $c_0 = 1$, this expansion is also point-normalized at $(0,0)$. Bivariate Taylor Interpolation Theorem 8.31 further implies that the point-normalized l'Hôpital dual asymptotic expansion s_1 Taylor interpolates the function f to order 0 on the lines $x = 0$ and $y = 0$ (a fact which we can also verify by direct inspection in such a simple example). The next step in the proof is to apply Bivariate Taylor Remainder Theorem 8.38 on the compact square $S_\rho := [-\rho, \rho]^2$ for an arbitrary real number $\rho > 0$. This requires us to calculate

$$\begin{aligned} (\partial_x^1 \partial_y^1 r_1)(x,y) &:= (D^2 \exp)(x+y) - (D \exp)(x)\,(D \exp)(y) \\ &= \exp(x+y) - \exp x \, \exp y \\ &=: r_1(x,y). \end{aligned}$$

Note that the previous equation states the following striking result:

$$\partial_x^1 \partial_y^1 r_1 = r_1.$$

At first glance, this would appear to present a problem for our analysis; however, we will shortly turn this problem into an opportunity to prove that the remainder r_1 does indeed vanish—using only theory previously discussed, we can show that r_1 vanishes locally, and upon further analysis using additional theory, we can show that r_1 also vanishes globally.

Since $|x|\,|y| \le \rho^2$ for all $(x, y) \in S_\rho$, we obtain the following error estimate from Bivariate Taylor Remainder Theorem 8.38 by letting $X := Y := [-\rho, \rho]$ and $x_0 := y_0 := 0$ and $n := 1$ and $r := r_1|S_\rho$:

$$\|r_1|S_\rho\|_\infty \le \left\|\partial_x^1 \partial_y^1 r_1|S_\rho\right\|_\infty \cdot \rho^2 = \|r_1|S_\rho\|_\infty \cdot \rho^2.$$

Let us rewrite the previous inequality as

$$0 \le \|r_1|S_\rho\|_\infty \cdot (\rho^2 - 1).$$

If $\rho < 1$, this result leads to a contradiction unless $\|r_1|S_\rho\|_\infty = 0$. We conclude that r_1 vanishes on the compact square S_ρ for all $0 < \rho < 1$. This implies that r_1 also vanishes on the open square

$$\bigcup_{0<\rho<1} S_\rho = (-1, 1)^2.$$

We can extend this local result to obtain a global result as follows: Since r_1 is real-analytic on the connected set $\mathbb{R}^2$, and since r_1 vanishes on the nonempty open subset $(-1, 1)^2$, we conclude that r_1 must vanish on all of $\mathbb{R}^2$ by the uniqueness of the analytic continuation; however, with some additional work, we can obtain this global result *without* assuming analyticity! This alternate approach is based on an adaptation of a standard technique that is used to obtain global results in the Volterra theory of integral equations; such integral equations arise naturally in connection with the Riemann function for second-order hyperbolic linear partial differential equations.[2]

We have shown that the remainder r_1 satisfies the following second-order hyperbolic linear partial differential equation with homogenous Cauchy data on the characteristic lines $x = 0$ and $y = 0$:

$$\partial_x^1 \partial_y^1 r_1 = r_1 \quad \text{and} \quad E_0 r_1 = 0 \quad \text{and} \quad E^0 r_1 = 0. \tag{9.3}$$

This kind of initial value problem is referred to as a **characteristic Cauchy problem** in [Eps62, pp. 56–57, Krieger]; in general, this terminology is applied to any initial value prob-

[2]The author is deeply indebted to Professor David Siegel for explaining how to adapt Volterra integral equation theory in order to obtain the desired global result under weaker hypotheses, as well as for his valuable guidance concerning the use of standard terminology in the theory of partial differential equations.

lem involving a second-order hyperbolic linear partial differential equation whose Cauchy data are prescribed on the two characteristic curves through a specified point. The following original terminology describes a *generalized* (higher-order) characteristic Cauchy problem which arises naturally when we use Taylor interpolation on two lines to prove tensor product identities:

Definition 9.1 *Let $X, Y \subset \mathbb{R}$ be intervals, let $x_0 \in X$ and $y_0 \in Y$, and assume that $r \in C^{(n,n)}(X \times Y)$, where n is a positive integer. If there exists a real constant λ such that r satisfies the linear partial differential equation*

$$\partial_x^n \partial_y^n r = \lambda \cdot r$$

on the rectangle $X \times Y$, we say that r ***is a hyperbolic eigenfunction of order n with eigenvalue*** *λ. If r is a hyperbolic eigenfunction of order n, and if $x = x_0$ and $y = y_0$ are zero lines of r of multiplicity at least n, we say that r* ***solves the hyperbolic eigenproblem of order n with homogeneous Cauchy data on $x = x_0$ and $y = y_0$****. We also refer to this as the* ***homogeneous hyperbolic eigenproblem****.*

For example, equation (9.3) asserts that the remainder r_1 is a hyperbolic eigenfunction of order one with eigenvalue $\lambda = 1$. Furthermore, r_1 solves the hyperbolic eigenproblem of order one with homogeneous Cauchy data on $x = 0$ and $y = 0$. Our earlier arguments based on local remainder theory and the uniqueness of the analytic continuation actually show that *any analytic solution of homogeneous hyperbolic eigenproblem (9.3) must vanish on all of $\mathbb{R}^2$*. This specific example illustrates a principle which also holds in general: *The homogeneous hyperbolic eigenproblem always has a unique solution—the zero function!* The following theorem shows that this uniqueness result holds not only in the function space $C^\omega(X \times Y)$, but in the strictly larger function space $C^{(n,n)}(X \times Y)$ as well:

Theorem 9.2 (Homogeneous Hyperbolic Eigenproblem) *Let $X, Y \subset \mathbb{R}$ be intervals, let $x_0 \in X$ and $y_0 \in Y$, and assume that $r \in C^{(n,n)}(X \times Y)$, where n is a positive integer. If r solves the hyperbolic eigenproblem of order n with homogeneous Cauchy data on $x = x_0$ and $y = y_0$, then r vanishes on $X \times Y$.*

Proof. Letting $f_1 := r$ and $f_2 := 0$ in Bivariate Taylor Remainder Theorem 8.38 yields

$$r(x,y) = \frac{1}{[(n-1)!]^2} \cdot \int_{x_0}^{x} \int_{y_0}^{y} (x-s)^{n-1} (y-t)^{n-1} (\partial_x^n \partial_y^n r)(s,t)\, dt\, ds$$

for all $(x, y) \in X \times Y$. By definition, there exists a constant $\lambda \in \mathbb{R}$ such that $\partial_x^n \partial_y^n r = \lambda \cdot r$ on $X \times Y$. If we apply the hyperbolic eigenfunction property to the integral form of the bivariate remainder, we obtain the following Volterra integral equation, which is satisfied by the function r at all points $(x, y) \in X \times Y$:

$$r(x,y) = \frac{\lambda}{[(n-1)!]^2} \cdot \int_{x_0}^{x} \int_{y_0}^{y} (x-s)^{n-1} \, (y-t)^{n-1} \, r(s,t) \, dt \, ds. \tag{9.4}$$

This integral equation automatically incorporates the assumption that r vanishes on the lines $x = x_0$ and $y = y_0$. We must show that r also vanishes at all points of $X \times Y$ which do not lie on either of these two lines.

Fix an arbitrary point $(x_1, y_1) \in [X \backslash \{x_0\}] \times [Y \backslash \{y_0\}]$. Let R denote the compact rectangle with vertices $\{(x_i, y_j) : 0 \le i, j \le 1\}$, and note that integral equation (9.4) holds on $R \subset X \times Y$. Let $\rho := |x_1 - x_0| \, |y_1 - y_0|$, and note that $|x - s| \, |y - t| \le \rho$ for all $(x, y), (s, t) \in R$.

There are at most four possible cases that we must consider depending on whether $x_0 < x_1$ or $x_0 > x_1$, and whether $y_0 < y_1$ or $y_0 > y_1$. In order to handle all four cases succinctly with a single argument, let us use the signum function (given by $\operatorname{sgn} x := x / |x|$ for all $x \in \mathbb{R} \backslash \{0\}$) to define the constants

$$\sigma := \operatorname{sgn}(x_1 - x_0) \quad \text{and} \quad \tau := \operatorname{sgn}(y_1 - y_0).$$

Note that

$$\sigma \cdot (x - x_0) = |x - x_0| \quad \text{and} \quad \tau \cdot (y - y_0) = |y - y_0| \quad \text{for all} \quad (x, y) \in X \times Y.$$

In all four cases, integral equation (9.4) yields the following pointwise estimate, which is valid for all $(x, y) \in R$:

$$|r(x,y)| \le \frac{|\lambda|}{[(n-1)!]^2} \, \sigma\tau \cdot \int_{x_0}^{x} \int_{y_0}^{y} |x-s|^{n-1} \, |y-t|^{n-1} \, |r(s,t)| \, dt \, ds.$$

The previous inequality in turn yields the pointwise estimate

$$|r(x,y)| \le M \sigma\tau \cdot \int_{x_0}^{x} \int_{y_0}^{y} |r(s,t)| \, dt \, ds \quad \text{for all} \quad (x,y) \in R \tag{9.5}$$

where

$$M := \frac{|\lambda|\,\rho^{n-1}}{[(n-1)!]^2} \geq 0.$$

We now arrive at the crux of the argument: Since the remainder r is continuous and the rectangle R is compact, it follows that

$$|r(x,y)| \leq \|r|R\|_\infty < \infty \quad \text{for all} \quad (x,y) \in R.$$

Applying this bound to inequality (9.5) yields the following pointwise estimate, which is valid for all $(x,y) \in R$:

$$|r(x,y)| \leq \|r|R\|_\infty \, M\sigma\tau \cdot \int_{x_0}^{x} \int_{y_0}^{y} 1 \, dt \, ds = \|r|R\|_\infty \, M\sigma\tau \cdot (x-x_0)(y-y_0).$$

Applying this new pointwise estimate to inequality (9.5) yields another pointwise estimate which is again valid for all $(x,y) \in R$:

$$\begin{aligned} |r(x,y)| &\leq \|r|R\|_\infty \, (M\sigma\tau)^2 \cdot \int_{x_0}^{x} \int_{y_0}^{y} (s-x_0)(t-y_0) \, dt \, ds \\ &= \|r|R\|_\infty \, (M\sigma\tau)^2 \cdot \frac{(x-x_0)^2}{2} \frac{(y-y_0)^2}{2}. \end{aligned}$$

Applying this newest pointwise estimate to inequality (9.5) yields yet another pointwise estimate which is once again valid for all $(x,y) \in R$:

$$\begin{aligned} |r(x,y)| &\leq \|r|R\|_\infty \, (M\sigma\tau)^3 \cdot \int_{x_0}^{x} \int_{y_0}^{y} \frac{(s-x_0)^2}{2} \frac{(t-y_0)^2}{2} \, dt \, ds \\ &= \|r|R\|_\infty \, (M\sigma\tau)^3 \cdot \frac{(x-x_0)^3}{3!} \frac{(y-y_0)^3}{3!}. \end{aligned}$$

Repeating this iterative process a total of m times yields the following pointwise estimate, which is valid for all positive integers m:

$$|r(x,y)| \leq \|r|R\|_\infty \, (M\sigma\tau)^m \cdot \frac{(x-x_0)^m}{m!} \frac{(y-y_0)^m}{m!} \quad \text{for all} \quad (x,y) \in R.$$

Letting $(x, y) := (x_1, y_1)$ in particular yields

$$\begin{aligned}
|r(x_1, y_1)| &\leq \|r|R\|_\infty (M\sigma\tau)^m \cdot \frac{(x_1 - x_0)^m}{m!} \frac{(y_1 - y_0)^m}{m!} \\
&= \|r|R\|_\infty \frac{M^m}{(m!)^2} \cdot |x_1 - x_0|^m \, |y_1 - y_0|^m \\
&= \|r|R\|_\infty \frac{(M\rho)^m}{(m!)^2} \quad \text{for all} \quad m \geq 1.
\end{aligned}$$

This final inequality and the limit

$$\lim_{m\to\infty} \frac{(M\rho)^m}{(m!)^2} = 0$$

together imply that $r(x_1, y_1) = 0$. Finally, since the point $(x_1, y_1) \in [X \backslash \{x_0\}] \times [Y \backslash \{y_0\}]$ was arbitrary, we conclude that r vanishes on the entire rectangle $X \times Y$. ■

We will use the previous theorem in the proof phase of all the derivation and proof algorithms developed in this chapter; the derivation phase, however, will be performed in three distinctly different ways—one way for each type of dual asymptotic expansion. We will explore each of these three original algorithms in turn in the next three sections.

9.2 Derivations via l'Hôpital Dual Asymptotic Expansions

In this section, we will assemble the methods of the previous section into a complete algorithm for the automatic derivation and proof of tensor product identities of a specified rank. The derivation phase of this algorithm is based on l'Hôpital dual asymptotic expansions, and the proof phase is based on the uniqueness of the solution to the homogeneous hyperbolic eigenproblem. The algorithm is as follows:

Algorithm 9.3 (l'Hôpital Identity) *Let $X, Y \subset \mathbb{R}$ be intervals, let $x_0 \in X$ and $y_0 \in Y$, and assume that $f \in C^{(n,n)}(X \times Y)$, where n is a given positive integer. If the following algorithm aborts with an error, then no conclusion can be drawn; however, if the algorithm terminates normally, then f satisfies a tensor product identity of rank n on $X \times Y$, and the algorithm will derive and prove this identity automatically:*

1. *Attempt to perform n iterations of l'Hôpital Dual Asymptotic Expansion Algorithm*

8.36 on f at the point (x_0, y_0) to determine whether $f \in \mathrm{LDAE}^n_{(x_0,y_0)}(X \times Y)$. If unsuccessful, then abort the algorithm with an error; however, if successful, then let s_n denote the resulting l'Hôpital dual asymptotic expansion of f to n terms at (x_0, y_0).

2. *Form the remainder $r_n := f - s_n$, which satisfies $r_n \in C^{(n,n)}(X \times Y)$. Bivariate Taylor Interpolation Theorem 8.31 guarantees that $x = x_0$ and $y = y_0$ are zero lines of r_n of multiplicity at least n.*

3. *Attempt to find a constant $\lambda \in \mathbb{R}$ such that $\partial_x^n \partial_y^n r_n = \lambda \cdot r_n$ on $X \times Y$. If unsuccessful, then abort the algorithm with an error; however, if successful, then r_n solves the hyperbolic eigenproblem of order n with homogeneous Cauchy data on $x = x_0$ and $y = y_0$. In that case, Homogeneous Hyperbolic Eigenproblem Theorem 9.2 guarantees that r_n vanishes on $X \times Y$.*

4. *Return the proven tensor product identity $f = s_n$.*

Of course, if $Y = X$ and the function f is *symmetric*, we can let $y_0 := x_0$ and replace l'Hôpital Dual Asymptotic Expansion Algorithm 8.36 with *Symmetric* l'Hôpital Dual Asymptotic Expansion Algorithm 8.37 in order to more efficiently derive and prove a *symmetric* tensor product identity. The following example illustrates this symmetric modification to l'Hôpital Identity Algorithm 9.3 by deriving and proving the addition formula for cosine:

Define $f \in C^\omega(\mathbb{R}^2)$ by $f(x, y) := \cos(x + y)$ for all $(x, y) \in \mathbb{R}^2$, and note that f is a symmetric function. Let us perform two iterations of Symmetric l'Hôpital Dual Asymptotic Expansion Algorithm 8.37 on f at $(0, 0)$. In the first iteration, we obtain

$$\begin{aligned}
g_0(x) &:= (E^0 \partial_y^0 r_0)(x) = (E^0 f)(x) = \cos(x + 0) = \cos x \\
c_0 &:= D^0 g_0(0) = \cos 0 = 1 \neq 0 \\
s_1(x, y) &:= s_0(x, y) + [g_0(x)\, g_0(y)]/c_0 = \cos x\, \cos y \\
r_1(x, y) &:= \cos(x + y) - \cos x\, \cos y.
\end{aligned}$$

In the second iteration, we obtain

$$\begin{aligned}
g_1(x) &:= (E^0 \partial_y^1 r_1)(x) = -\sin(x + 0) + \cos x\, \sin 0 = -\sin x \\
c_1 &:= D^1 g_1(0) = -\cos 0 = -1 \neq 0 \\
s_2(x, y) &:= s_1(x, y) + [g_1(x)\, g_1(y)]/c_1 = \cos x\, \cos y - \sin x\, \sin y \\
r_2(x, y) &:= \cos(x + y) - \cos x\, \cos y + \sin x\, \sin y.
\end{aligned}$$

This shows that $f \in \mathrm{LDAE}^2_{(0,0)}(\mathbb{R}^2)$. The subsequent calculation

$$\begin{aligned}(\partial_x^1 \partial_y^1 r_2)(x,y) &= -\cos(x+y) - \sin x\, \sin y + \cos x\, \cos y \\ (\partial_x^2 \partial_y^2 r_2)(x,y) &= \cos(x+y) - \cos x\, \cos y + \sin x\, \sin y =: r_2(x,y)\end{aligned}$$

reveals that r_2 is a hyperbolic eigenfunction of order two with eigenvalue $\lambda = 1$. By the symmetric version of l'Hôpital Identity Algorithm 9.3, we conclude that f satisfies a symmetric tensor product identity of rank two, namely

$$\cos(x+y) = \cos x\, \cos y - \sin x\, \sin y \quad \text{for all} \quad (x,y) \in \mathbb{R}^2.$$

If we apply the same technique to the symmetric function $f(x,y) := \sin(x+y)$ at the point $(0,0)$, all we will manage to show is that $f \notin \mathrm{LDAE}^1_{(0,0)}(\mathbb{R}^2)$. The point $(0,0)$ is not suitable for expanding f in a l'Hôpital dual asymptotic expansion because $f(0,0) := \sin 0 = 0$. Do not despair, dear reader—all is not lost! The alternate point $(0, \frac{\pi}{2})$ is perfectly suitable since $f \in \mathrm{LDAE}^2_{(0,\pi/2)}(\mathbb{R}^2)$; however, we will have to work a little harder since we can no longer apply the *symmetric* version of l'Hôpital Identity Algorithm 9.3. Applying the *original* version of this algorithm to f at $(0, \frac{\pi}{2})$ will derive and prove the following tensor product identity of rank two:

$$\sin(x+y) = \sin(x + \frac{\pi}{2})\, \sin y - \cos(x + \frac{\pi}{2})\, \cos y \quad \text{for all} \quad (x,y) \in \mathbb{R}^2.$$

By substituting the special values $x := \frac{\pi}{2}$ and $y := 0$, we can use this identity to reduce itself to the standard symmetric form

$$\sin(x+y) = \cos x\, \sin y + \sin x\, \cos y \quad \text{for all} \quad (x,y) \in \mathbb{R}^2.$$

Another approach which works quite well is to expand the function $f(x,y) := \sin(x+y)$ in a l'Hôpital dual asymptotic expansion to two terms at the point $(0,\varepsilon)$, where $\varepsilon \neq 0$ is a small parameter. *This yields an intermediate identity which is valid for all* $(x,y) \in \mathbb{R}^2$ *and all* ε *sufficiently close to zero*. Taking the limit of this intermediate identity as $\varepsilon \to 0$ immediately yields the addition formula for sine in standard symmetric form.

A third approach—which is really the method of choice—is to derive the addition formulas for cosine and sine *simultaneously*. We can accomplish this by expanding the function of *three* variables

$$f(x, y, z) := \cos(x + y) + z \sin(x + y)$$

in a *symmetric* l'Hôpital dual asymptotic expansion to two terms *in the first two variables x and y* at the point $(0, 0)$. *This yields an intermediate identity which is valid for all $(x, y, z) \in \mathbb{R}^3$ and which depends on z in a polynomial fashion.* Equating like powers of z on both sides of this intermediate identity immediately yields the addition formulas for both cosine and sine! We can also derive and prove the addition formulas for hyperbolic cosine and hyperbolic sine—simultaneously—by applying exactly the same approach to the function

$$f(x, y, z) := \cosh(x + y) + z \sinh(x + y).$$

For any positive integer n, the previous approach extends quite nicely to the **generalized hyperbolic functions of order** n, which are the n functions $\{H_i\}_{i=0}^{n-1} \subset C^\omega(\mathbb{R})$ defined by the initial value problem $DH_i = H_{i-1}$ and $H_i(0) := \delta_{i0}$ for $0 \leq i \leq n - 1$, where the index $i - 1$ is interpreted modulo n, and δ_{ij} denotes the Kronecker delta.[3] We can derive addition formulas *for all n generalized hyperbolic functions simultaneously* by expanding the function of $n + 2$ variables

$$f(x, y, z_0, \ldots, z_{n-1}) := \sum_{i=0}^{n-1} z_i \cdot H_i(x + y)$$

in a symmetric l'Hôpital dual asymptotic expansion to n terms in the first two variables x and y at the point $(0, 0)$. *The resulting intermediate identity will be valid for all points $(x, y, z_0, \ldots, z_{n-1}) \in \mathbb{R}^{n+2}$, and both sides of the identity will be linear homogenous polynomials in the variables $z_0, \ldots, z_{n-1}$.* Equating the coefficients of z_i for $0 \leq i \leq n - 1$ yields

[3]The generalized hyperbolic functions are a fascinating study in their own right. They share strong connections with circulant matrices and Vandermonde determinants, which in turn are related to the discrete Fourier transform and the complex roots of unity. The generalized hyperbolic functions also describe a one-parameter subgroup of a commutative Lie group. The author's earliest research showed that this commutative Lie group describes a group of isometries in a non-Riemannian conformal geometry for hypercomplex function theory over the group algebra of the cyclic group. All of this richly interdisciplinary algebra, geometry, and analysis flows naturally from a simple (and often rediscovered) generalization of the hyperbolic functions!

addition formulas for all n functions $\{H_i\}_{i=0}^{n-1}$ at once! In fact, for $0 \le i \le n-1$, we obtain

$$H_i(x+y) = \sum_{j=0}^{n-1} H_j(x)\, H_{i-j}(y) \quad \text{for all} \quad (x,y) \in \mathbb{R}^2,$$

where the index $i-j$ is interpreted modulo n.

This completes our discussion of the first of three original derivation and proof algorithms for tensor product identities. We will discuss the remaining two algorithms in the next two sections.

9.3 Derivations via Strong Dual Asymptotic Expansions

Although most of the well-known examples of tensor product identities can be derived using *l'Hôpital* dual asymptotic expansions, not all such identities can be obtained by a direct application of this technique; we can, however, accommodate the exceptional cases by using *strong* dual asymptotic expansions instead. The following algorithm uses this alternate approach:

Algorithm 9.4 (Symmetric Strong Identity) *Let $a \in [-\infty, \infty)$, and define the unbounded open interval $X := (a, \infty)$. Let $x_0 \in X$, and define the unbounded open subinterval $X_0 := (x_0, \infty)$. Note that x_0 and ∞ are both limit points of X_0. Let $f \in C^{(n,n)}(X^2)$ be a symmetric function, where n is a given positive integer, and assume that $f|X_0^2 \in \mathrm{LNV}_{\infty}^{x_0}(X_0^2)$. If the following algorithm aborts with an error, then no conclusion can be drawn; however, if the algorithm terminates normally, then f satisfies a symmetric tensor product identity of rank n on X^2, and the algorithm will derive and prove this identity automatically:*

1. *Attempt to perform n iterations of Strong Dual Asymptotic Expansion Algorithm 7.33 on f at the point (∞, x_0) to determine whether $f \in \mathrm{SDAE}^{\,n}_{(\infty,x_0)}(X_0^2)$. If unsuccessful, then abort the algorithm with an error; however, if successful, then let s_n denote the resulting strong dual asymptotic expansion of f to n terms at (∞, x_0), and note that s_n is defined on the square X_0^2.*

2. *Attempt to extend s_n to a symmetric function $\tilde{s}_n \in C^{(n,n)}(X^2)$. If no such extension exists, then abort the algorithm with an error.*

3. *For all $x \in X$, determine whether the x-section $(\tilde{s}_n)_x$ is a l'Hôpital asymptotic expansion of the x-section f_x at the point $x_0 \in X$. If not, then abort the algorithm with an error. If so, then Univariate Taylor Interpolation Theorem 8.8 guarantees that $\tilde{s}_n$ Taylor interpolates f to order $n-1$ on $y = x_0$. By symmetry, $\tilde{s}_n$ also Taylor interpolates f to order $n-1$ on $x = x_0$.*

4. *Form the remainder $\tilde{r}_n := f - \tilde{s}_n$, which satisfies $\tilde{r}_n \in C^{(n,n)}(X^2)$, and note that $x = x_0$ and $y = x_0$ are zero lines of $\tilde{r}_n$ of multiplicity at least n.*

5. *Attempt to find a constant $\lambda \in \mathbb{R}$ such that $\partial_x^n \partial_y^n \tilde{r}_n = \lambda \cdot \tilde{r}_n$ on X^2. If unsuccessful, then abort the algorithm with an error; however, if successful, then $\tilde{r}_n$ solves the hyperbolic eigenproblem of order n with homogeneous Cauchy data on $x = x_0$ and $y = x_0$. In that case, Homogeneous Hyperbolic Eigenproblem Theorem 9.2 guarantees that $\tilde{r}_n$ vanishes on X^2.*

6. *Return the proven symmetric tensor product identity $f = \tilde{s}_n$.*

The main idea in this algorithm is to generate a strong dual asymptotic expansion at the point (∞, x_0) and then verify that the expansion is l'Hôpital in the variable y at the point x_0 for all fixed $x \in X$. This is sufficient to establish the Taylor interpolation properties of the expansion on the line $y = x_0$. By exploiting symmetry, we automatically obtain the Taylor interpolation properties of the expansion on the line $x = x_0$, which ensures that the remainder has homogeneous Cauchy data on both $x = x_0$ and $y = x_0$. Verifying the hyperbolic eigenfunction property of the remainder completes the proof. We note in particular that unlike l'Hôpital Identity Algorithm 9.3, where the use of symmetry is optional, Symmetric Strong Identity Algorithm 9.4 uses symmetry in an essential way to establish the Taylor interpolation properties on *both* of the rectilinear coordinate lines through the point (x_0, x_0).

Recall that at the end of Chapter 7, we noted there was one technical difficulty which typically arises when using Strong Dual Asymptotic Expansion Algorithm 7.33 in actual practice. In the context of Symmetric Strong Identity Algorithm 9.4, the difficulty is to verify the hypothesis

$$r_i \in \mathrm{LNV}_{\infty}^{x_0}(X_0^2).$$

The good news is that the Taylor interpolation properties of the extended partial sum $\tilde{s}_i$ on the line $y = x_0$ give us the means to overcome this difficulty easily. The analysis is as follows:

Recall that $X_0 := (x_0, \infty)$, and fix $x \in X_0$. Assume that $1 \le i \le n-1$. Since the x-section $f_x \in C^n(X)$ Taylor interpolates the x-section $(\tilde{s}_i)_x \in C^n(X)$ to order $i-1$ at $x_0 \in X$, Univariate Taylor Remainder Theorem 8.10 implies that for every $x_1 \in X_0$, there exists some $\xi \in X_0$ such that the x-section $(\tilde{r}_i)_x := f_x - (\tilde{s}_i)_x$ satisfies

$$(\tilde{r}_i)_x(x_1) = \frac{D^i(\tilde{r}_i)_x(\xi)}{i!} \cdot (x_1 - x_0)^i.$$

Since $\tilde{r}_i|X_0^2 = r_i$ by definition, and since $x, x_1, \xi \in X_0$, we can drop the tildes in the previous equation. When this mean-value remainder formula is expressed without using the cross-section notation, it asserts that for each $(x, x_1) \in X_0^2$, there is some $\xi \in X_0$ such that

$$r_i(x, x_1) = \frac{\partial_y^i r_i(x, \xi)}{i!} \cdot (x_1 - x_0)^i.$$

Now assume the following additional hypothesis, which takes the form of a differential inequality:

$$\partial_y^i r_i > 0 \quad \text{on} \quad X_0^2.$$

Since $x_1 \in X_0$ means that $x_1 > x_0$, it follows that $(x_1 - x_0)^i > 0$. The above mean-value remainder formula therefore implies that

$$r_i > 0 \quad \text{on} \quad X_0^2.$$

We conclude that

$$r_i \in \mathrm{GNV}(X_0^2) \subset \mathrm{LNV}_\infty^{x_0}(X_0^2).$$

Remark 9.5 *The above analysis shows that if the differential inequality $\partial_y^i r_i > 0$ holds on X_0^2 in the context of Symmetric Strong Identity Algorithm 9.4, then $r_i \in \mathrm{LNV}_\infty^{x_0}(X_0^2)$. This sufficient condition is easy to apply in actual practice since the partial derivative $\partial_y^i r_i$ is generally much simpler than the original remainder r_i in typical applications of the algorithm.*

We have specified Symmetric Strong Identity Algorithm 9.4 in a modular form, partly for the sake of simplicity, and partly in order to clearly indicate the connections with previously established theory. In actual practice, we must integrate all of the analytical techniques just discussed into one seamless whole. The following example demonstrates this seamless integration by deriving and proving the multiplication formula for the natural logarithm function using strong dual asymptotic expansions:

Define the unbounded open interval $X := (0, \infty)$ and the unbounded open subinterval $X_0 := (1, \infty)$. Let $f \in C^\omega(X^2)$ by given by $f(x, y) := \log(xy)$ for all $(x, y) \in X^2$, and note that f is a symmetric function. Since $f > 0$ on X_0^2, it follows that $f|X_0^2 \in \mathrm{GNV}(X_0^2) \subset \mathrm{LNV}^1_\infty(X_0^2)$. Let us perform two iterations of Strong Dual Asymptotic Expansion Algorithm 7.33 on the function f at the point $(\infty, 1)$. In the first iteration, we obtain the following results, which hold for all $x, y, x', y' \in X_0$:

$$
\begin{aligned}
(r_0 \circ_\infty r_0)(y, y') &:= \lim_{x\to\infty} \frac{f(x, y')}{f(x, y)} = \lim_{x\to\infty} \frac{\log(xy')}{\log(xy)} = \lim_{x\to\infty} \frac{y'/(xy')}{y/(xy)} = \frac{1}{1} \\
h_0(y) &:= 1 > 0 \\
(r_0 \circ^1 r_0)(x', x) &:= \lim_{y\to 1^+} \frac{f(x', y)}{f(x, y)} = \lim_{y\to 1^+} \frac{\log(x'y)}{\log(xy)} = \frac{\log x'}{\log x} \\
g_0(x) &:= \log x > 0 \\
L_{01} &:= \lim_{x\to\infty} \left(\lim_{y\to 1^+} \frac{r_0(x, y)}{g_0(x)\, h_0(y)} \right) = \lim_{x\to\infty} \left(\lim_{y\to 1^+} \frac{\log(xy)}{\log x} \right) = \lim_{x\to\infty} \frac{\log x}{\log x} = 1 \\
L_{02} &:= \lim_{y\to 1^+} \left(\lim_{x\to\infty} \frac{r_0(x, y)}{g_0(x)\, h_0(y)} \right) = \lim_{y\to 1^+} \left(\lim_{x\to\infty} \frac{\log(xy)}{\log x} \right) \\
&= \lim_{y\to 1^+} \left(\lim_{x\to\infty} \frac{y/(xy)}{1/x} \right) = \lim_{y\to 1^+} 1 = 1 \\
c_0 &:= \lim_{\{x\to\infty,\ y\to 1^+\}} \frac{r_0(x, y)}{g_0(x)\, h_0(y)} = L_{01} = L_{02} = 1 \neq 0 \\
s_1(x, y) &:= s_0(x, y) + c_0 \cdot g_0(x)\, h_0(y) = \log x \\
r_1(x, y) &:= \log(xy) - \log x.
\end{aligned}
$$

Clearly, we can extend the partial sum s_1 and remainder r_1 to functions $\tilde{s}_1$ and $\tilde{r}_1$ which are analytic on the entire first quadrant X^2. We note in particular that

$$\tilde{r}_1(x, 1) = \log x - \log x = 0 \quad \text{for all} \quad x \in X,$$

which shows that $y = 1$ is a zero line of $\tilde{r}_1$ of multiplicity at least one. The differential inequality

$$\partial_y^1 r_1(x, y) := \frac{x}{xy} - 0 = \frac{1}{y} > 0 \quad \text{for all} \quad (x, y) \in X_0^2$$

further implies that

$$r_1 > 0 \quad \text{on} \quad X_0^2,$$

and thus

$$r_1 \in \mathrm{GNV}(X_0^2) \subset \mathrm{LNV}_\infty^1(X_0^2).$$

This last condition assures us that we can continue on to the second iteration of Strong Dual Asymptotic Expansion Algorithm 7.33.

Note that the other partial derivative of the remainder r_1 satisfies

$$\partial_x^1 r_1(x,y) := \frac{y}{xy} - \frac{1}{x} = 0 \quad \text{for all} \quad (x,y) \in X_0^2.$$

This tells us that r_1 is a function of y alone. We could use this fact to finish the derivation and proof directly by a specialized argument; however, in order to illustrate a much more widely applicable method, we will use the fact that r_1 is independent of x simply to evaluate the otherwise intractable limit

$$\lim_{x\to\infty} [\log(xy) - \log x] = \lim_{x\to 1^+} [\log(xy) - \log x] = \log y - \log 1 = \log y \quad \text{for all} \quad y \in X_0.$$

This limit occurs several times in the second iteration of the algorithm. In the second iteration, we obtain the following results, which hold for all $x, y, x', y' \in X_0$:

$$
\begin{aligned}
(r_1 \circ_\infty r_1)(y, y') &:= \lim_{x\to\infty} \frac{\log(xy') - \log x}{\log(xy) - \log x} = \frac{\lim\limits_{x\to\infty}[\log(xy') - \log x]}{\lim\limits_{x\to\infty}[\log(xy) - \log x]} = \frac{\log y'}{\log y} \\
h_1(y) &:= \log y > 0 \\
(r_1 \circ^1 r_1)(x', x) &:= \lim_{y\to 1^+} \frac{\log(x'y) - \log x'}{\log(xy) - \log x} = \lim_{y\to 1^+} \frac{x'/(x'y) - 0}{x/(xy) - 0} = \frac{1}{1} \\
g_1(x) &:= 1 > 0 \\
L_{11} &:= \lim_{x\to\infty}\left(\lim_{y\to 1^+} \frac{r_1(x,y)}{g_1(x)\, h_1(y)}\right) = \lim_{x\to\infty}\left(\lim_{y\to 1^+} \frac{\log(xy) - \log x}{\log y}\right) \\
&= \lim_{x\to\infty}\left(\lim_{y\to 1^+} \frac{x/(xy) - 0}{1/y}\right) = \lim_{x\to\infty} 1 = 1 \\
L_{12} &:= \lim_{y\to 1^+}\left(\lim_{x\to\infty} \frac{r_1(x,y)}{g_1(x)\, h_1(y)}\right) = \lim_{y\to 1^+}\left(\lim_{x\to\infty} \frac{\log(xy) - \log x}{\log y}\right) \\
&= \lim_{y\to 1^+}\left(\frac{\lim_{x\to\infty}[\log(xy) - \log x]}{\log y}\right) = \lim_{y\to 1^+} \frac{\log y}{\log y} = 1 \\
c_1 &:= \lim_{\{x\to\infty,\ y\to 1^+\}} \frac{r_1(x,y)}{g_1(x)\, h_1(y)} = L_{11} = L_{12} = 1 \neq 0 \\
s_2(x,y) &:= s_1(x,y) + c_1 \cdot g_1(x)\, h_1(y) = \log x + \log y \\
r_2(x,y) &:= \log(xy) - \log x - \log y.
\end{aligned}
$$

This shows that $f \in \mathrm{SDAE}^2_{(\infty,1)}(X_0^2)$.

Clearly, we can extend the partial sum s_2 and remainder r_2 to *symmetric* functions $\tilde{s}_2$ and $\tilde{r}_2$ which are analytic on the entire first quadrant X^2. Strong Dual Asymptotic Expansion Algorithm 7.33 assures us that for each fixed parameter $x \in X_0$, the following univariate asymptotic expansion holds:

$$\log(xy) \sim \log x \cdot 1 + 1 \cdot \log y \quad \text{as} \quad y \to 1^+.$$

By Proposition 6.17, this univariate asymptotic expansion holds for all $x \in X_0$ if and only if the following limits hold for all $x \in X_0$:

$$\log x = \lim_{y \to 1^+} \frac{\log(xy)}{1} \quad \text{and} \quad 1 = \lim_{y \to 1^+} \frac{\log(xy) - \log x \cdot 1}{\log y}.$$

Since these limits are preserved under the extension process—in other words, the limits actually hold for all $x \in X$—the univariate asymptotic expansion above also holds for all $x \in X$. In addition, this univariate expansion is l'Hôpital in y at the point $1 \in X$ since the asymptotic sequence consisting of 1 and $\log y$ is l'Hôpital (and point-normalized) at 1. This ensures that $y = 1$ is a zero line of $\tilde{r}_2$ of multiplicity at least two, and by symmetry, $x = 1$ is a zero line of $\tilde{r}_2$ of multiplicity at least two as well.

The calculation

$$\partial_y^1 \tilde{r}_2(x, y) = \frac{x}{xy} - 0 - \frac{1}{y} = 0 \quad \text{for all} \quad (x, y) \in X^2$$

suffices to show that $\tilde{r}_2$ is a hyperbolic eigenfunction of order two with eigenvalue $\lambda = 0$. (Actually, it shows that $\tilde{r}_2$ is a hyperbolic eigenfunction of all positive orders with eigenvalue $\lambda = 0$). By Symmetric Strong Identity Algorithm 9.4, we conclude that f satisfies a symmetric tensor product identity of rank two, namely

$$\log(xy) = \log x + \log y \quad \text{for all} \quad (x, y) \in X^2.$$

Although this may seem like a rather complicated way to derive and prove a rather simple identity, this observation in and of itself has great value: It calls our attention to various ways in which we might further refine the underlying theory in order to substantially simplify both Symmetric Strong Identity Algorithm 9.4 and the ensuing proof of the multiplication formula for the natural logarithm function.

Let us briefly consider another well-known example—the binomial theorem, which asserts the following for all $n \in \mathbb{N}$:

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot x^{n-i} \, y^i \quad \text{for all} \quad (x, y) \in \mathbb{R}^2.$$

The binomial theorem is a symmetric tensor product identity of rank $n + 1$, valid on the entire plane $\mathbb{R}^2$. Can we derive and prove this identity using the methods of this section?

If we restrict the identity to the first quadrant $(0,\infty)^2$, the right-hand side becomes a strong dual asymptotic expansion of the left-hand side to $n+1$ terms at the point $(\infty, 0)$. For each fixed $x > 0$, the resulting univariate asymptotic expansion is l'Hôpital in the variable y at 0 since the asymptotic sequence $\{y^i\}_{i=0}^n$ is l'Hôpital at 0.

Furthermore, since the derivatives[4] satisfy $D^i y^j = 0$ for $i > j$, the extended partial sums $\{\tilde{s}_i\}_{i=0}^{n+1}$ satisfy

$$\partial_y^i \tilde{s}_i(x,y) = \sum_{j=0}^{i-1} \binom{n}{j} \cdot x^{n-j} \left(D^i y^j\right) = 0$$

for all $(x,y) \in \mathbb{R}^2$. As a result, the restricted remainders $\{r_i\}_{i=0}^n$ satisfy the differential inequality

$$\partial_y^i r_i(x,y) = \partial_y^i (x+y)^n - \partial_y^i s_i(x,y) = \frac{n!}{(n-i)!} \cdot (x+y)^{n-i} > 0$$

for all $(x,y) \in (0,\infty)^2$, which tells us that

$$r_i \in \mathrm{GNV}\left((0,\infty)^2\right) \subset \mathrm{LNV}_\infty^0\left((0,\infty)^2\right).$$

Finally, we note that

$$\partial_y^{n+1} \tilde{r}_{n+1}(x,y) = \partial_y^{n+1}(x+y)^n - \partial_y^{n+1} \tilde{s}_{n+1}(x,y) = 0 \quad \text{for all} \quad (x,y) \in \mathbb{R}^2,$$

which tell us that the extended remainder $\tilde{r}_{n+1}$ is a hyperbolic eigenfunction of order $n+1$ with eigenvalue $\lambda = 0$.

On the basis of the above analysis, we conclude that Symmetric Strong Identity Algorithm 9.4—the same algorithm we used to derive and prove the multiplication formula for the natural logarithm function—can also be used to derive and prove the binomial theorem![5] Who ever would have guessed that a single algorithm comprised of such ele-

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[4]In calculating the derivatives of polynomials here, are we *assuming* the very knowledge that we seek to prove? Although the binomial theorem is sometimes used to *prove* the formula $Dx^n = n \cdot x^{n-1}$ for all $n \in \mathbb{N}$, this can also be accomplished without invoking any algebraic identities: We can use the definition of the derivative to prove the result quite easily for $n = 0$ and $n = 1$, and then use the definition $x^{n+1} = x \cdot x^n$ together with the product rule to prove the general result by induction on n. The proposed derivation and proof of the binomial theorem therefore stand on solid ground.

[5]In the logarithm example, we used a specialized technique to evaluate an otherwise intractable limit as $x \to \infty$. In the binomial example, no such specialized technique is needed—all the limits of quotients as $x \to \infty$ can be evaluated by a combination of standard techniques involving simple algebra and l'Hôpital's rule. By the way, the intermediate calculations in these two examples would make very interesting calculus problems for first-year undergraduates!

mentary operations could actually derive and prove identities *for both transcendental and polynomial functions?*

This completes our discussion of the second of three original derivation and proof algorithms for tensor product identities. In the next section, we will turn our attention to the third and final algorithm.

9.4 Derivations via Weak Dual Asymptotic Expansions

The first two algorithms of this chapter share a common design philosophy which can be described as follows: *Use as much asymptotic theory and interpolation theory as possible in order to minimize the need for direct calculation.* For example, l'Hôpital Identity Algorithm 9.3 exploited the Taylor interpolation properties of l'Hôpital dual asymptotic expansions, and Symmetric Strong Identity Algorithm 9.4 exploited the Taylor interpolation properties of l'Hôpital univariate asymptotic expansions in conjunction with symmetry in order to achieve exactly the same end. In both cases, the established Taylor interpolation properties guaranteed that the final remainder had homogeneous Cauchy data on the two characteristic lines—and thus satisfied two of the three main conditions which define the homogeneous hyperbolic eigenproblem. In effect, we obtained two thirds of the input to the proof phase from the output of the derivation phase—at virtually no cost!

Here is a complementary philosophy which also has a legitimate place in the design of algorithms: *Minimize the theoretical prerequisites for the sake of simplicity and then compensate for the missing theory by doing more direct calculation.* This approach is especially suitable when someone else is available to do all of the hard work! In the Dark Ages, such menial tasks were often relegated to a special kind of highly-skilled servant known as a "research assistant"; however, since we now live in the Age of Enlightenment, we can relegate these tedious tasks to our tireless electronic servant, the desktop computer—properly equipped with a powerful computer algebra system such as Maple, of course! The following algorithm embraces this complementary philosophy of algorithm design:

Algorithm 9.6 (Weak Identity) *Let $X, Y \subset \mathbb{R}$ be intervals. For $i \in \{0, 1\}$, let x_i be a limit point of X, let y_i be a limit point of Y, and let n_i be a positive integer. Note that x_0 and x_1 may or may not be distinct; the same applies to y_0 and y_1, and to n_0 and n_1 as well. In addition, assume that $x_1 \in X$, that $y_1 \in Y$, and that $f \in C^{(n_1,n_1)}(X \times Y)$. If the* *following algorithm aborts with an error, then no conclusion can be drawn; however, if the algorithm terminates normally, then f satisfies a tensor product identity of rank at most n_0 on $X \times Y$, and the algorithm will derive and prove this identity automatically:*

1. *Attempt to perform n_0 iterations of Weak Dual Asymptotic Expansion Algorithm 7.29 on f at the point (x_0, y_0). If unsuccessful, then abort the algorithm with an error; however, if successful, then let s_{n_0} denote the result. If $s_{n_0} \notin C^{n_1}(X) \otimes C^{n_1}(Y)$, then abort the algorithm with an error.*

2. *Form the remainder $r_{n_0} := f - s_{n_0}$, which satisfies $r_{n_0} \in C^{(n_1,n_1)}(X \times Y)$. Determine by direct calculation whether $x = x_1$ and $y = y_1$ are zero lines of r_{n_0} of multiplicity at least n_1. If not, then abort the algorithm with an error.*

3. *Attempt to find a constant $\lambda \in \mathbb{R}$ such that $\partial_x^{n_1} \partial_y^{n_1} r_{n_0} = \lambda \cdot r_{n_0}$ on $X \times Y$. If unsuccessful, then abort the algorithm with an error; however, if successful, then r_{n_0} solves the hyperbolic eigenproblem of order n_1 with homogeneous Cauchy data on $x = x_1$ and $y = y_1$. In that case, Homogeneous Hyperbolic Eigenproblem Theorem 9.2 guarantees that r_{n_0} vanishes on $X \times Y$.*

4. *Return the proven tensor product identity $f = s_{n_0}$.*

A comparison of the three algorithms of this chapter reveals a definite trend towards increasing generality:

1. l'Hôpital Identity Algorithm 9.3 generated a l'Hôpital dual asymptotic expansion to n terms at the point $(x_0, y_0) \in X \times Y$. The algorithm then established that the remainder r_n satisfied the hyperbolic eigenproblem of *order also* n with homogeneous Cauchy data on the two characteristic lines through the *same point* (x_0, y_0).

2. Symmetric Strong Identity Algorithm 9.4 generated a strong dual asymptotic expansion to n terms at the point $(\infty, x_0) \in \bar{X} \times X$. The algorithm then established that the remainder r_n satisfied the hyperbolic eigenproblem of *order also* n with homogeneous Cauchy data on the two characteristic lines through a *different point* (x_0, x_0).

3. Weak Identity Algorithm 9.6 generated a tensor product with n_0 terms using the expansion point $(x_0, y_0) \in \bar{X} \times \bar{Y}$. The algorithm then established that the remainder r_{n_0} satisfied the hyperbolic eigenproblem of *(possibly) different order* n_1 with homogeneous Cauchy data on the two characteristic lines through a *(possibly) different point* (x_1, y_1).

Increased generality provides greater flexibility in applications. Since every l'Hôpital dual asymptotic expansion and every strong dual asymptotic expansion are both weak dual asymptotic expansions as well, every tensor product identity which we can derive and prove by the first two algorithms can also be accommodated by the third algorithm; the converse is false, however, since there are some identities which actually require the greater generality of the third algorithm.

For example, if $X := Y := \mathbb{R}$, only the third algorithm is sufficiently general to accommodate the expansion point $(\infty, \infty) \in \overline{\mathbb{R}}^2$. This fact is of genuine interest because the expansion point (∞, ∞) can be used to derive the Christoffel-Darboux formula for orthogonal polynomials,[6] which asserts that

$$\frac{k_n}{k_{n+1}\, h_n} \cdot \frac{p_{n+1}(x)\, p_n(y) - p_n(x)\, p_{n+1}(y)}{x - y} = \sum_{i=0}^{n} \frac{1}{h_{n-i}} \cdot p_{n-i}(x)\, p_{n-i}(y)$$

for all $(x, y) \in \mathbb{R}^2$. (The line of singularities $y = x$ on the left-hand side can be removed by taking the limit as $y \to x$ and applying l'Hôpital's rule.) In this formula, p_i denotes an orthogonal polynomial of degree i, the positive constant h_i is a normalization constant for p_i, and the nonzero constant k_i is the leading coefficient of p_i.

The Christoffel-Darboux formula is a symmetric tensor product identity of rank $n + 1$ on the plane $\mathbb{R}^2$. Since *any* degree-descending polynomial sequence $\{p_{n-i}(x)\}_{i=0}^{n}$ is *automatically* an asymptotic sequence as $x \to \infty$, the right-hand side of the formula (with the order of summation as written above) is a *weak dual asymptotic expansion* of the left-hand side to $n + 1$ terms at the point (∞, ∞). Under the specified normalization, Uniqueness Theorem 7.27 for weak dual asymptotic expansions implies that the subset of n orthogonal polynomials $\{p_i\}_{i=0}^{n-1}$ appearing on the right-hand side is *uniquely determined* by the *two* orthogonal polynomials p_n and p_{n+1} on the left-hand side! This observation is consistent with the fact that each family of classical orthogonal polynomials satisfies a *second-order* linear recurrence relation (Table 22.7 in [AS64, p. 782, Dover]).

Note that instead of performing the proof phase on the remainder $r_{n+1}(x, y)$, which

[6] The Christoffel-Darboux formula (Equation 22.12.1 in [AS64, p. 785, Dover]) holds for *every* family of classical orthogonal polynomials: the Jacobi polynomials, which are orthogonal on the interval $(-1, 1)$ with respect to the weight function $(1 - x)^\alpha (1 + x)^\beta$ for $\alpha, \beta > -1$; important special cases of the Jacobi polynomials, such as the Legendre polynomials ($\alpha := \beta := 0$), the Chebyshev polynomials of the first and second kinds ($\alpha =: \beta := \mp\frac{1}{2}$), and the Gegenbauer polynomials ($\alpha := \beta := \lambda - \frac{1}{2}$ for $\lambda > -\frac{1}{2}$); the Laguerre polynomials, which are orthogonal on $(0, \infty)$ with respect to the weight function $x^\alpha \exp(-x)$ for $\alpha > -1$; and the Hermite polynomials, which are orthogonal on $(-\infty, \infty)$ with respect to the weight function $\exp(-x^2)$.

ostensibly involves a rational function, it is much simpler to perform the proof phase on the modified remainder $(x - y) \cdot r_{n+1}(x, y)$, which is unequivocally polynomial. Proving that $(x - y) \cdot r_{n+1}(x, y)$ vanishes on $\mathbb{R}^2$ implies that $r_{n+1}(x, y)$ vanishes on all of $\mathbb{R}^2$ except possibly on the line $y = x$, but by continuity, $r_{n+1}(x, y)$ must vanish on $y = x$ as well. (Of course, the use of modified remainders requires a minor enhancement to Weak Identity Algorithm 9.6, but we can easily accommodate this change.)

Embracing a design which minimizes the use of theory and maximizes the use of direct calculation makes Weak Identity Algorithm 9.6 particularly well-suited for implementation in a computer algebra system. The author has implemented a slightly simplified version of Weak Identity Algorithm 9.6 in Version 8 of Maple, thereby creating a complete derivation and proof system for tensor product identities—all in less than 50 lines of Maple code! In keeping with the two-phase algorithm design used throughout this chapter, this Maple implementation consists of two routines: `derive` and `prove`.

In his recently accepted ISSAC conference paper [Cha03], the author successfully applied his `derive` and `prove` routines to six different classes of functions: exponential and logarithm functions, trigonometric functions, hyperbolic functions, generalized hyperbolic functions, polynomial functions, and polynomial-exponential functions. To establish the universality of these techniques, the conference paper demonstrated how to use this Maple implementation to derive and prove *25 different tensor product identities*—including *all* of the classical tensor product identities generally known by first-year university undergraduates!

The Maple source code for the `derive` and `prove` routines has been included in the thesis in Appendix A, *Using Maple to Derive and Prove Identities*. This appendix also includes examples which illustrate how to use Maple to derive and prove all of the identities that we derived and proved by hand earlier in this chapter.

This completes our study of algorithms for the automatic derivation and proof of tensor product identities for smooth functions on rectangles. It may interest the reader to know that the author's general program of research continues this computational motif by using tensor products to develop algorithms for a closely-related purpose: the uniform approximation of continuous functions on compact rectangles. If we cannot make the approximation error vanish—as in the case of tensor product identities—at least we can make it vanishingly small!

Chapter 10

Natural Tensor Product Interpolation

The thesis has studied two interpolation problems in depth: interpolation by natural tensor products on the lines of a two-dimensional grid, and Taylor interpolation by natural tensor products on two rectilinear coordinate lines. This chapter of the thesis will develop some abstractions which both unify and generalize these two interpolation problems and their exact series solutions. Along the way, we will also codify many standard and original results in the fundamental theory of natural tensor product interpolation—doing so at an unprecedented level of generality! This level of generality in turn provides a solid and flexible foundation for future applications of this work.

The methods of this chapter draw heavily upon both concrete matrix algebra and abstract linear algebra. On the concrete side, we will develop formulas for the determinant and inverse of matrices partitioned into blocks. On the abstract side, we will develop a precise original notion of commuting linear functionals over linear configurations of vector spaces, and we will use this notion to define the *abstract* splitting operator in full generality. We will then extend this notion to *families* of commuting linear functions in order to specify both weak and strong interpolation problems. We will also give the first rigorous definition in the mathematical literature of the space of *natural* tensor products. We will conclude this chapter by using our partitioned matrix formulas to develop an iterative algorithm which uses the abstract splitting operator to generate exact series solutions to the strong interpolation problem in the space of natural tensor products.

10.1 A Brief Historical Overview and Assessment

The idea of interpolation—and approximation—by natural tensor products is by no means new. This section presents a brief historical overview and assessment of prior research in this area. To facilitate this overview and assessment, we will find it particularly helpful to articulate three levels of abstraction for specifying problems in natural tensor product interpolation:

1. The **low level of abstraction** is based on concrete function spaces and uses concrete linear functionals to specify the data to be interpolated. These functions spaces could be spaces such as $F^{X \times Y}$ or $C^{(n,n)}(X \times Y)$, for example, and the linear functionals could be evaluation functionals such as ε_{x_i} and ε^{y_i}, or differentiation-evaluation functionals such as $\varepsilon_{x_0} D^i$ and $\varepsilon^{y_0} D^i$. In this context, it is understood that all the data to be interpolated on the bivariate function space are derived using *parametric extensions* of the specified linear functionals on univariate function spaces. Interpolation on the lines of a two-dimensional grid is one example of a natural tensor product interpolation problem at this low level of abstraction, and Taylor interpolation on two rectilinear coordinate lines is another.

2. The **intermediate level of abstraction** is also based on concrete functions spaces, but uses abstract linear functionals to specify the data to be interpolated. For example, we might still work over a function space like $C^{(n,n)}(X \times Y)$, but we would use arbitrary linear functionals $\phi \in C^n(X)^*$ and $\psi \in C^n(Y)^*$ to specify the desired interpolation data. Since we are still working over function spaces, we can once again *parametrically extend* the linear functionals ϕ and ψ to linear operators Φ and Ψ on the bivariate function space $C^{(n,n)}(X \times Y)$.

3. The **high level of abstraction** is based on abstract vector spaces and uses abstract linear functionals to specify the data to be interpolated. At this level of abstraction, we work over an abstract vector space W containing the tensor product $U \otimes V$ of two abstract vector spaces, and we use arbitrary linear functionals $\phi \in U^*$ and $\psi \in V^*$ to specify the desired interpolation data. Since we are no longer working over function spaces, we must postulate some kind of *abstract extension process* which allows us to extend the linear functionals ϕ and ψ to linear operators Φ and Ψ on W.

Let us now use these levels of abstraction to assess the contributions of various researchers in the field of natural tensor product interpolation.

Harry Bateman, in his 1922 paper [Bat22], applied natural tensor product interpolation to the kernels of Fredholm linear integral equations of the second kind in order to develop a practical indirect method for the approximate solution of such integral equations. Bateman's method is based on the problem of interpolation on the lines of a two-dimensional grid, and thus represents a *low level of abstraction*. Bateman specified the natural tensor product solution to this interpolation problem *implicitly* using an equation involving a determinant. Bateman offered his indirect method for solving Fredholm integral equations as a serviceable alternative to Erhard Schmidt's more costly indirect method of 1907, which was based on determining the *best mean-square approximation* to a square-integrable kernel by a tensor product [Sch07a], [Sch07b].

Charles Micchelli and Allan Pinkus, in their 1977 paper [MP77], applied natural tensor product interpolation to solve the problem of determining the *best mean approximation* to a continuous kernel by a tensor product—assuming that the kernel belongs to the class of strictly totally positive functions. Like Bateman, Micchelli and Pinkus also worked at a *low level of abstraction*, and based their approach on the underlying problem of interpolation on the lines of a two-dimensional grid; however, Micchelli and Pinkus specified the natural tensor product solution to this interpolation problem *explicitly* using determinants and cofactors. They also stated—without proof—a very useful formula which expresses the associated remainder in closed form as the *ratio of two determinants*. The main result of Micchelli and Pinkus on best mean approximation was considered noteworthy by Ward Cheney, who cited their result in his concise 1986 survey of multivariate approximation theory [Che86, p. 23].

The phrase "natural tensor product" was actually coined by Donald Thomas in his 1976 paper [Tho76]. This paper was important because it provided the first systematic development of natural tensor product interpolation theory *at the intermediate level of abstraction*. Indeed, Thomas's paper made several fundamental contributions to this area of research:

1. Thomas formulated natural tensor product interpolation theory in greater generality than other researchers by using *abstract linear functionals* on concrete function spaces. He also demonstrated the usefulness of this increased generality by presenting a wider variety of interpolation problems—including an example involving *Hermite interpolation* on the lines of a two-dimensional grid.

2. Thomas solved the *general* natural tensor product interpolation problem at the intermediate level of abstraction. He specified an *explicit* solution which was expressed

succinctly and elegantly *in the language of matrix algebra* using the pseudoinverse of a rectangular matrix. After seeing this, the explicit solution of Micchelli and Pinkus—expressed in terms of determinants and cofactors—seems a little cumbersome in comparison!

3. Thomas described and briefly illustrated a *general technique* for using existing *univariate remainder theories* to develop corresponding *bivariate remainder theories* for problems in natural tensor product interpolation.

Earlier in the thesis, the author adapted the remainder method of Thomas and used his technique in the proof of Bivariate Taylor Remainder Theorem 8.38. In a public lecture presented at the University of Waterloo in December 2002, the author outlined the proof of a comparable remainder theorem for the problem of natural tensor product interpolation on the lines of a two-dimensional grid; this result was also proven by adapting the method of Thomas. In another public lecture presented at Lehigh University in September 2001, the author presented a preliminary version of the remainder theory for this same problem. This earlier lecture did *not* use the method of Thomas because at that time, the author was not convinced that the method was correct—unfortunately, Thomas did not include full details in his illustrative example, and failed to explain how to overcome a potential difficulty that one would naturally anticipate when using two successive applications of univariate mean-value remainder formulas to develop a bivariate mean-value remainder formula.

The author is now fully satisfied that the remainder method of Thomas is correct, and has already illustrated this method in complete detail—for both integral and mean-value forms of the remainder—in the proof of Bivariate Taylor Remainder Theorem 8.38. In addition, using the method of Thomas in 2002 resulted in an enormous simplification of the author's remainder formula of 2001 for the problem of natural tensor product interpolation on the lines of a two-dimensional grid. The author now wholeheartedly embraces the method of Thomas, and offers this summary of a lesson learned the hard way:

Remark 10.1 *The remainder method of Thomas is a truly labor-saving device! Please, dear reader, do the whole world a humongous favor by using the method of Thomas whenever humanly possible!!*[1]

Returning to our historical overview, we note in summary that natural tensor products were applied—without being called by that name—by Bateman in 1922 and by Micchelli

[1] The view from this soapbox is really quite breathtaking!

and Pinkus in 1977 to develop techniques for approximating continuous kernels of integral equations; these approximation methods used the cross-sections of a given kernel to construct a natural tensor product which performed ordinary interpolation on the lines of a two-dimensional grid. Thomas substantially extended the subject in 1976 by developing a framework sufficiently general to accommodate not only ordinary interpolation on grid lines, but also Taylor interpolation on two lines, and even full Hermite interpolation on grid lines—as well as Schmidt's aforementioned Hilbert-space technique based on the truncation of the orthogonal eigenfunction expansion of a square-integrable kernel!

The remainder of this chapter codifies the fundamental theory of natural tensor product interpolation by collecting together the elementary results (but not the more advanced applications) which are scattered throughout the papers of Bateman, Micchelli and Pinkus, and Thomas. The thesis will actually surpass these earlier works by providing the first systematic development of the theory *at the high level of abstraction.*

Note that Thomas defined the unique natural tensor product *interpolant*, but did not define the *vector space* of natural tensor products to which the interpolant belongs; this strikes the author as akin to defining the unique polynomial interpolant without first explaining what it means for a function to be a polynomial. The thesis will remedy this omission by providing the mathematical literature's first formal definition of *natural* tensor products as a *subspace* of the space of tensor products. The thesis will also state and prove the ratio-of-determinants remainder formula of Micchelli and Pinkus—but will do so for the *general* natural tensor product interpolation problem *at the high level of abstraction.* We will conclude this original exposition of the abstract theory by proving that the series expansions generated by the iterative algorithms of the thesis are in fact natural tensor product interpolants in the sense discussed in the mathematical literature.

10.2 Determinants and Inverses of Partitioned Matrices

As we noted in the previous section, Donald Thomas made a major contribution to natural tensor product interpolation in his 1976 paper [Tho76] by formulating the theory in the language of matrix algebra. Continuing in the research direction initiated by Thomas, the thesis will now advance the theory further by demonstrating how standard formulas for matrices partitioned into blocks can be applied with great success to develop the fundamental theorems of natural tensor product interpolation. This original contribution of the author is based on well-established material such as can be found in [HJ85, pp. 18–19, 21–22].

Let V be a vector space over a field F, and n be a positive integer. We will denote the vector space of all n-dimensional row vectors with elements in V by $\mathrm{Row}_n(V)$, the vector space of all n-dimensional column vectors with elements in V by $\mathrm{Col}_n(V)$, and the vector space of all $n \times n$ matrices with elements in V by $\mathrm{Mat}_n(V)$. Although our immediate needs will be quite adequately served by the special case $V = F$, we will need the full generality of these definitions later in this chapter. For example, we may let V be an abstract vector space, a dual space, or even a space of linear operators!

Remark 10.2 *As a conceptual aid, we will adopt the convention of denoting the elements of* $\mathrm{Row}_n(V)$ *and* $\mathrm{Col}_n(V)$ *and* $\mathrm{Mat}_n(V)$ *by boldface letters when* $V \neq F$ *(to indicate that the elements are vectors) and by ordinary letters when* $V = F$ *(to indicate that the elements are scalars). For example,* $\mathbf{v} \in \mathrm{Row}_n(V)$ *denotes a row vector whose elements are vectors, whereas* $v \in \mathrm{Row}_n(F)$ *denotes a row vector whose elements are scalars.*

We will now discuss some standard formulas for the determinants and inverses of matrices partitioned into blocks. Instead of developing these formulas in full generality, we will develop them in a simpler special case which is sufficient for our work with natural tensor products. Let $M \in \mathrm{Mat}_n(F)$ denote a matrix, $r \in \mathrm{Row}_n(F)$ denote a row, $c \in \mathrm{Col}_n(F)$ denote a column, and $s \in F$ denote a scalar. Define the partitioned matrix $P \in \mathrm{Mat}_{n+1}(F)$ by

$$P := \begin{bmatrix} M & c \\ r & s \end{bmatrix}. \tag{10.1}$$

If M is invertible, we can define the **Schur complement of** M **in** P, denoted by $\bar{M}$, via

$$\bar{M} := s - rM^{-1}c. \tag{10.2}$$

Note that $\bar{M} \in F$. If s is invertible, we can define the **Schur complement of** s **in** P, denoted by $\bar{s}$, via

$$\bar{s} := M - cs^{-1}r. \tag{10.3}$$

Note that $\bar{s} \in \mathrm{Mat}_n(F)$. The convention of denoting the Schur complement with an overbar is due to the author; we will find that this convention dramatically simplifies all of our subsequent formulas.

If the matrix block M is invertible, the Schur complement $\bar{M}$ is well-defined, and the

Schur determinant formula for the partitioned matrix P asserts that

$$\det P = \det M \cdot \det \bar{M}. \tag{10.4}$$

This formula can be derived and proven by applying the determinant to the following product of partitioned matrices and exploiting the indicated upper-triangular and lower-triangular block structures:

$$\begin{bmatrix} M & c \\ r & s \end{bmatrix} \cdot \begin{bmatrix} I & -M^{-1}c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} M & 0 \\ r & \bar{M} \end{bmatrix}.$$

The Schur determinant formula implies that P is invertible whenever both M and $\bar{M}$ are invertible. Under these hypotheses, we will now develop an explicit formula for P^{-1} which is expressed succinctly in terms of the block structure of the partitioned matrix P.

Assume that both s and M are invertible, which ensures that both Schur complements $\bar{s}$ and $\bar{M}$ are well-defined. Under these hypotheses, the Schur determinant formula

$$\det P = \det M \cdot \det \bar{M} = \det s \cdot \det \bar{s}$$

implies that $\bar{s}$ is invertible whenever $\bar{M}$ is invertible.

Note that the definition $\bar{s} := M - cs^{-1}r$ expresses the Schur complement $\bar{s}$ as the sum of an invertible matrix M and a matrix $-cs^{-1}r$ of rank at most one. A standard formula for the inverse of a rank-one adjustment to an invertible matrix asserts that

$$(\bar{s})^{-1} := M^{-1} + (M^{-1}crM^{-1})/\bar{M}. \tag{10.5}$$

This formula also expresses $(\bar{s})^{-1}$ as the sum of an invertible matrix M^{-1} and a matrix $(M^{-1}crM^{-1})/\bar{M}$ of rank at most one.

A direct calculation proves that equations (10.3) and (10.5) indeed satisfy $\bar{s} \cdot (\bar{s})^{-1} = I$. The key to simplifying the intermediate results is to obtain a common denominator consisting of the scalar $s \cdot \bar{M}$, to apply the definition of $\bar{M}$ given in equation (10.2) to the resulting numerator, and finally to use the relation $(crM^{-1})^2 = (rM^{-1}c) \cdot crM^{-1}$, which holds since $rM^{-1}c$ is a scalar and therefore commutes with matrices.[2]

Assume that s, M, and $\bar{M}$, are all invertible, which as we noted above implies that $\bar{s}$ is

[2] Verifying the formula for $(\bar{s})^{-1}$ is a good exercise in matrix algebra, highly recommended to keen students everywhere!

invertible also. Under these hypotheses, a standard formula for the inverse of a partitioned matrix asserts that

$$P^{-1} = \begin{bmatrix} (\bar{s})^{-1} & -(M^{-1}c)/\bar{M} \\ -(rM^{-1})/\bar{M} & 1/\bar{M} \end{bmatrix}. \tag{10.6}$$

Substituting equation (10.5) into equation (10.6) yields

$$P^{-1} = \begin{bmatrix} M^{-1} + (M^{-1}crM^{-1})/\bar{M} & -(M^{-1}c)/\bar{M} \\ -(rM^{-1})/\bar{M} & 1/\bar{M} \end{bmatrix}, \tag{10.7}$$

which is well-defined whenever both M and $\bar{M}$ are invertible; recall that Schur determinant formula (10.4) implies that these hypotheses are sufficient to ensure the invertibility of P. Note that equation (10.7) is now completely independent of $\bar{s}$, and actually holds under weaker hypotheses than equation (10.6); for example, equation (10.7) holds even if $s = 0$.

A direct calculation proves that equations (10.1) and (10.7) indeed satisfy $P \cdot P^{-1} = I$. The key to simplifying the intermediate results is to obtain a common denominator consisting of the scalar $\bar{M}$, and to apply the definition of $\bar{M}$ given in equation (10.2) to the resulting numerator.[3]

With these formulas for the determinant and the inverse of a partitioned matrix at our disposal, we are ready to take the next step in developing the fundamental theory of natural tensor product interpolation: We must create a proper context for this work, and will do so using abstract linear functionals on abstract vector spaces in the next section.

10.3 Linear Configurations and Commuting Functionals

Remember that our ultimate goal is to construct an element of an abstract tensor product space $U \otimes V$ which interpolates prescribed linear functional data for a given element of an abstract vector space W. The following original definition describes the structure that the triple (U, V, W) of abstract vector spaces must possess in order to be suitable for this kind of interpolation problem:

Definition 10.3 *Let F be a field. We say that the triple (U, V, W)* ***is a linear configuration over*** *F if all three of the following conditions are satisfied:*

[3] Verifying the second, more general formula for P^{-1} is also a good exercise in matrix algebra, and is actually easier than verifying the formula for $(\bar{s})^{-1}$.

1. *U, V, and W are vector spaces over F.*

2. *$U \cap V = F$.*

3. *$U \otimes V \subset W$.*

This simple definition has many implications which are essential for natural tensor product interpolation. The hypothesis $U \cap V = F$ implies that $F \subset U$ and $F \subset V$. From this, we obtain

$$F = F \otimes F \subset U \otimes V,$$

which implies that $F \subset U \otimes V$ as well. Similarly,

$$U = U \otimes F \subset U \otimes V \quad \text{and} \quad V = F \otimes V \subset U \otimes V,$$

which implies that $U \subset U \otimes V$ and $V \subset U \otimes V$. The hypothesis $U \otimes V \subset W$ further implies that $F \subset W$, that $U \subset W$, and that $V \subset W$ also. In summary, if (U, V, W) is a linear configuration over F, it follows that the field F is a subspace of U, V, $U \otimes V$, and W; that U and V are subspaces of $U \otimes V$; and that U, V, and $U \otimes V$ are all subspaces of W.

The following original definition describes the properties that two linear functionals $\phi \in U^*$ and $\psi \in V^*$ must possess in order to prescribe interpolation data in a manner that is suitable for our work:

Definition 10.4 *Let (U, V, W) be a linear configuration over a field F. We say that the linear functionals $\phi \in U^*$* ***and*** *$\psi \in V^*$* ***commute over*** *W if the derived linear maps*

$$\phi \otimes I \in \mathrm{Hom}_F(U \otimes V,\ V) \quad \textit{and} \quad I \otimes \psi \in \mathrm{Hom}_F(U \otimes V,\ U)$$

can be extended to linear maps

$$\Phi \in \mathrm{Hom}_F(W, V) \quad \textit{and} \quad \Psi \in \mathrm{Hom}_F(W, U)$$

with the property

$$\Phi\Psi = \Psi\Phi.$$

Let $U^\bullet \subset U^*$ and $V^\bullet \subset V^*$ be arbitrary subsets. We say that the families of linear functionals $U^\bullet$ ***and*** *$V^\bullet$* ***commute over*** *W if $\phi \in U^*$ and $\psi \in V^*$ commute over W for all $(\phi, \psi) \in U^\bullet \times V^\bullet$.*

This simple definition also has many implications which are essential for natural tensor product interpolation. By the definition of the extensions,

$$\Phi|(U \otimes V) = \phi \otimes I \quad \text{and} \quad \Psi|(U \otimes V) = I \otimes \psi.$$

Since the derived maps satisfy

$$(\phi \otimes I)|U = \phi \quad \text{and} \quad (I \otimes \psi)|V = \psi,$$

it follows that we can recover the original linear functionals ϕ and ψ from the linear maps Φ and Ψ by restriction:

$$\Phi|U = \phi \quad \text{and} \quad \Psi|V = \psi.$$

Furthermore, the commutativity of the extensions Φ and Ψ implies that

$$\operatorname{ran}(\Phi\Psi) = \operatorname{ran}(\Psi\Phi) \subset \operatorname{ran}\Psi \cap \operatorname{ran}\Phi \subset U \cap V = F.$$

We conclude that the commuting composition $\Phi\Psi$ is actually a linear functional on W:

$$\Phi\Psi \in W^*.$$

Finally, note that we can recover the derived linear functional $\phi \otimes \psi \in (U \otimes V)^*$ from the linear functional $\Phi\Psi \in W^*$ by restriction:

$$(\Phi\Psi)|(U \otimes V) = \phi \otimes \psi.$$

This completes our development of the postulated extension process for abstract linear functions on abstract vector spaces. We will use this abstract extension process to define the abstract splitting operator in the next section.

10.4 The Abstract Splitting Operator

The next step in our program for natural tensor product interpolation is to define the abstract splitting operator in terms of commuting linear functionals over a linear configuration of abstract vector spaces:

Definition 10.5 *Let (U, V, W) be a linear configuration over a field F, and assume that* *the linear functionals $\phi \in U^*$ and $\psi \in V^*$ commute over W. Let Φ and Ψ denote commuting extensions of $\phi \otimes I$ and $I \otimes \psi$ from $U \otimes V$ to W, respectively. Define the following subsets of W and $U \otimes V$:*

$$W_{(\phi,\psi)} := W \setminus \ker(\Phi\Psi) \quad \text{and} \quad (U \otimes V)_{(\phi,\psi)} := (U \otimes V) \setminus \ker(\phi \otimes \psi).$$

*The **abstract splitting operator induced by** (ϕ, ψ) is the nonlinear map*

$$\Omega_{(\phi,\psi)} : W_{(\phi,\psi)} \to (U \otimes V)_{(\phi,\psi)}$$

defined by

$$\Omega_{(\phi,\psi)} w := \frac{(\Psi \otimes \Phi)(w \otimes w)}{(\Phi\Psi)(w)} \quad \text{for all} \quad w \in W_{(\phi,\psi)}.$$

Let us consider the implications of this definition. Recalling the definition of $\Psi \otimes \Phi$, we obtain

$$\Omega_{(\phi,\psi)} w := \frac{(\Psi \otimes \Phi)(w \otimes w)}{(\Phi\Psi)(w)} = \frac{(\Psi w) \otimes (\Phi w)}{(\Phi\Psi)(w)},$$

which means that the image $\Omega_{(\phi,\psi)} w$ is a dyad in $U \otimes V$. Moreover, the hypothesis

$$(\Phi\Psi)(w) = (\Psi\Phi)(w) \neq 0$$

implies that

$$\Psi w \neq 0 \quad \text{and} \quad \Phi w \neq 0,$$

which means that $\Omega_{(\phi,\psi)} w$ is a tensor product of rank one. Applying the linear functional $\phi \otimes \psi$ to the rank-one tensor product $\Omega_{(\phi,\psi)} w$, we obtain

$$(\phi \otimes \psi)(\Omega_{(\phi,\psi)} w) = \frac{(\phi\Psi w) \otimes (\psi\Phi w)}{(\Phi\Psi)(w)} = \frac{(\Phi\Psi)(w) \cdot (\Psi\Phi)(w)}{(\Phi\Psi)(w)} = (\Psi\Phi)(w) \neq 0,$$

which implies that

$$\Omega_{(\phi,\psi)} w \in (U \otimes V)_{(\phi,\psi)},$$

as claimed. Proceeding in a similar fashion, we can easily show that the operator $\Omega_{(\phi,\psi)}$ is homogeneous with respect to nonzero scalars, and that the operator $\Omega_{(\phi,\psi)}$ is idempotent as well.

Let us now consider two examples which will illustrate all of the abstract formalisms introduced in the chapter thus far. In addition, the ensuing discussion of these examples will establish important connections between the newly defined *abstract* splitting operator and previously studied special cases of the *asymptotic* splitting operator.

Example 10.6 (Evaluation Functionals) *Let X and Y be arbitrary sets, and let F be a field. Define the function spaces $U := F^X$ and $V := F^Y$ and $W := F^{X\times Y}$. Since $U \cap V = F$ and $U \otimes V \subset W$, the triple (U, V, W) is a linear configuration over F by Definition 10.3. Choose $x_0 \in X$ and $y_0 \in Y$. Let $\varepsilon_{x_0} \in U^*$ and $\varepsilon^{y_0} \in V^*$ denote evaluation functionals, and let $E_{x_0} \in \mathrm{Hom}_F(W, V)$ and $E^{y_0} \in \mathrm{Hom}_F(W, U)$ denote the corresponding evaluation operators. By considering how E_{x_0} and E^{y_0} transform dyads in $U \otimes V$, we can easily show that $E_{x_0}|(U \otimes V) = \varepsilon_{x_0} \otimes I$ and $E^{y_0}|(U \otimes V) = I \otimes \varepsilon^{y_0}$. This means that E_{x_0} and E^{y_0} extend $\varepsilon_{x_0} \otimes I$ and $I \otimes \varepsilon^{y_0}$ from $U \otimes V$ to W. Since $E_{x_0}E^{y_0} = E^{y_0}E_{x_0}$, we conclude that the linear functionals ε_{x_0} and ε^{y_0} commute over W by Definition 10.4. For convenience, let $\phi := \varepsilon_{x_0}$ and $\psi := \varepsilon^{y_0}$, and let $\Phi := E_{x_0}$ and $\Psi := E^{y_0}$, following the notational conventions of Section 3.2. Let us also invoke the notation of Section 5.1 and define the subsets*

$$\begin{aligned} W_{(\phi,\psi)} &:= W \setminus \ker(\Phi\Psi) = \{f \in W : f(x_0, y_0) \neq 0\} =: W_{(x_0,y_0)} \\ (U \otimes V)_{(\phi,\psi)} &:= (U \otimes V) \setminus \ker(\phi \otimes \psi) = \{t \in U \otimes V : t(x_0, y_0) \neq 0\} =: (U \otimes V)_{(x_0,y_0)}. \end{aligned}$$

By Definition 10.5, the abstract splitting operator induced by (ϕ, ψ) is the nonlinear map

$$\Omega_{(\phi,\psi)} : W_{(\phi,\psi)} \to (U \otimes V)_{(\phi,\psi)}$$

given for all $f \in W_{(\phi,\psi)}$ by

$$\Omega_{(\phi,\psi)} f := \frac{(\Psi \otimes \Phi)(f \otimes f)}{(\Phi\Psi)(f)} = \frac{(\Psi f) \otimes (\Phi f)}{(\Phi\Psi)(f)} = \frac{(E^{y_0} f) \otimes (E_{x_0} f)}{(E_{x_0} E^{y_0})(f)}.$$

How is this example of the *abstract* splitting operator related to the *asymptotic* splitting operator? If we evaluate the previous equation at an arbitrary point $(x, y) \in X \times Y$, we obtain

$$\Omega_{(\phi,\psi)} f(x, y) = \frac{(E^{y_0} f)(x)\,(E_{x_0} f)(y)}{(E_{x_0} E^{y_0})(f)} = \frac{f(x, y_0)\, f(x_0, y)}{f(x_0, y_0)} = \Upsilon_{(x_0,y_0)} f(x, y).$$

This shows that the *abstract* splitting operator $\Omega_{(\phi,\psi)} : W_{(\phi,\psi)} \to (U \otimes V)_{(\phi,\psi)}$ induced by the evaluation functionals $\phi := \varepsilon_{x_0}$ and $\psi := \varepsilon^{y_0}$ is identical to the special case of the *asymptotic* splitting operator $\Upsilon_{(x_0,y_0)} : W_{(x_0,y_0)} \to (U \otimes V)_{(x_0,y_0)}$ which we originally defined in Section 5.1 and later revisited in Section 5.3.

Example 10.7 (Normal Derivatives) *Let $X, Y \subset \mathbb{R}$ be intervals, and let n be either a positive integer or infinity. Define the function spaces $U := C^n(X)$ and $V := C^n(Y)$ and $W := C^{(n,n)}(X \times Y)$. Since $U \cap V = \mathbb{R}$ and $U \otimes V \subset W$, the triple (U, V, W) is a linear configuration over $\mathbb{R}$ by Definition 10.3. Choose $x_0 \in X$ and $y_0 \in Y$. For $0 \leq i < n$, define the linear functionals $\phi_i := \varepsilon_{x_0} D_x^i \in U^*$ and $\psi_i := \varepsilon^{y_0} D_y^i \in V^*$. Their parametric extensions $\Phi_i := E_{x_0} \partial_x^i \in \mathrm{Hom}_{\mathbb{R}}(W, V)$ and $\Psi_i := E^{y_0} \partial_y^i \in \mathrm{Hom}_{\mathbb{R}}(W, U)$ are the i-th order normal derivative operators on the lines $x = x_0$ and $y = y_0$. By considering how Φ_i and Ψ_i transform dyads in $U \otimes V$, we can easily show that $\Phi_i|(U \otimes V) = \phi_i \otimes I$ and $\Psi_i|(U \otimes V) = I \otimes \psi_i$. This means that Φ_i and Ψ_i extend $\phi_i \otimes I$ and $I \otimes \psi_i$ from $U \otimes V$ to W. Since $\Phi_i \Psi_i = \Psi_i \Phi_i$ by Clairaut's theorem, we conclude that the linear functionals ϕ_i and ψ_i commute over W by Definition 10.4. Let us define the subsets*

$$\begin{aligned} W_{(\phi_i,\psi_i)} &:= W \backslash \ker(\Phi_i \Psi_i) = \{f \in W : (\partial_x^i \partial_y^i f)(x_0, y_0) \neq 0\} \\ (U \otimes V)_{(\phi_i,\psi_i)} &:= (U \otimes V) \backslash \ker(\phi_i \otimes \psi_i) = \{t \in U \otimes V : (\partial_x^i \partial_y^i t)(x_0, y_0) \neq 0\}. \end{aligned}$$

By Definition 10.5, the abstract splitting operator induced by (ϕ_i, ψ_i) is the nonlinear map

$$\Omega_{(\phi_i,\psi_i)} : W_{(\phi_i,\psi_i)} \to (U \otimes V)_{(\phi_i,\psi_i)}$$

given for all $r_i \in W_{(\phi_i,\psi_i)}$ by

$$\Omega_{(\phi_i,\psi_i)} r_i := \frac{(\Psi_i \otimes \Phi_i)(r_i \otimes r_i)}{(\Phi_i \Psi_i)(r_i)} = \frac{(\Psi_i r_i) \otimes (\Phi_i r_i)}{(\Phi_i \Psi_i)(r_i)} = \frac{(E^{y_0} \partial_y^i r_i) \otimes (E_{x_0} \partial_x^i r_i)}{(E_{x_0} \partial_x^i)(E^{y_0} \partial_y^i)(r_i)}.$$

Recall that we defined the asymptotic splitting operator $\Upsilon_{(x_0,y_0)}$ in full generality in Section 7.2, and later revisited $\Upsilon_{(x_0,y_0)}$ in the context of l'Hôpital dual asymptotic expansions in Subsection 8.3.2. In keeping with that context, let us now assume that $x = x_0$ and $y = y_0$ are zero lines of $r_i \in W_{(\phi_i,\psi_i)}$ of multiplicity i. If we evaluate the previous equation at an arbitrary point $(x, y) \in X \times Y$ under this assumption, we obtain

$$\Omega_{(\phi,\psi_i)} r_i(x, y) = \frac{(E^{y_0} \partial_y^i r_i)(x)\,(E_{x_0} \partial_x^i r_i)(y)}{(E_{x_0} \partial_x^i)(E^{y_0} \partial_y^i)(r_i)} = \frac{(\partial_y^i r_i)(x, y_0)\,(\partial_x^i r_i)(x_0, y)}{(\partial_x^i \partial_y^i r_i)(x_0, y_0)} = \Upsilon_{(x_0,y_0)} r_i(x, y).$$

The final equality follows from the general definition of $\Upsilon_{(x_0,y_0)}$ by invoking l'Hôpital's rule a total of $2i$ times (i times in each variable separately). Under these hypotheses, we conclude that the *abstract* splitting operator $\Omega_{(\phi_i,\psi_i)}$ induced by the linear functionals $\phi_i := \varepsilon_{x_0} D_x^i$ and $\psi_i := \varepsilon^{y_0} D_y^i$ coincides with the special case of the *asymptotic* splitting operator $\Upsilon_{(x_0,y_0)}$ used to generate the i-th term of a l'Hôpital dual asymptotic expansion at (x_0, y_0).

10.5 Strong and Weak Interpolation Problems

Let us now consider the interpolation properties of the abstract splitting operator $\Omega_{(\phi,\psi)}$ induced by two linear functionals $\phi \in U^*$ and $\psi \in V^*$ which commute over W. Let Φ and Ψ denote commuting extensions of $\phi \otimes I$ and $I \otimes \psi$ from $U \otimes V$ to W. Let $w \in W_{(\phi,\psi)}$, and recall that $\Omega_{(\phi,\psi)} w \in U \otimes V$. Since $\Phi|(U \otimes V) = \phi \otimes I$ and $\Phi|U = \phi$, applying the operator Φ to the tensor product $\Omega_{(\phi,\psi)} w$ yields

$$\Phi\Omega_{(\phi,\psi)} w = \frac{(\phi \otimes I)(\Psi w \otimes \Phi w)}{\Phi\Psi w} = \frac{(\phi\Psi w) \otimes (\Phi w)}{\Phi\Psi w} = \frac{(\Phi\Psi w) \cdot (\Phi w)}{\Phi\Psi w} = \Phi w.$$

Recall that $\Phi\Psi = \Psi\Phi$. Since $\Psi|(U \otimes V) = I \otimes \psi$ and $\Psi|V = \psi$, applying the operator Ψ to the tensor product $\Omega_{(\phi,\psi)} w$ yields

$$\Psi\Omega_{(\phi,\psi)} w = \frac{(I \otimes \psi)(\Psi w \otimes \Phi w)}{\Psi\Phi w} = \frac{(\Psi w) \otimes (\psi\Phi w)}{\Psi\Phi w} = \frac{(\Psi\Phi w) \cdot (\Psi w)}{\Psi\Phi w} = \Psi w.$$

We have shown that

$$\Phi\Omega_{(\phi,\psi)} w = \Phi w \quad \text{and} \quad \Psi\Omega_{(\phi,\psi)} w = \Psi w \quad \text{for all} \quad w \in W_{(\phi,\psi)}.$$

This means that the tensor product $\Omega_{(\phi,\psi)} w$ and the vector w generate the same data under the action of the linear extensions Φ and Ψ of the linear functionals ϕ and ψ. As a special case of this result, we also obtain

$$\Phi\Psi\Omega_{(\phi,\psi)} w = \Phi\Psi w \quad \text{for all} \quad w \in W_{(\phi,\psi)}.$$

This means that the tensor product $\Omega_{(\phi,\psi)} w$ and the vector w generate the same data under the action of the linear functional $\Phi\Psi$. The tensor product $\Omega_{(\phi,\psi)} w$ thus *interpolates* the vector w in both a *strong* sense (the actions of Φ and Ψ separately) and a *weak* sense (the combined action of $\Phi\Psi$).

The strong and weak interpolation problems which are solved by the abstract splitting operator $\Omega_{(\phi,\psi)}$ have something in common: Both problems are determined by the commuting pair of linear functionals $(\phi, \psi) \in U^* \times V^*$. The following original definition formalizes this notion of strong and weak interpolation problems over an abstract vector space W and extends it to higher orders:

Definition 10.8 *Let (U, V, W) be a linear configuration over a field F. Let n be a positive integer, and let $\{\phi_i\}_{i=0}^{n-1} \subset U^*$ and $\{\psi_i\}_{i=0}^{n-1} \subset V^*$ be arbitrary linear functionals. We say that*

$$\Pi_n := (\phi_0, \phi_1, \ldots, \phi_{n-1}; \psi_0, \psi_1, \ldots, \psi_{n-1}) \in (U^*)^n \times (V^*)^n$$

***is an interpolation problem of order** n **over** (U, V, W) if all three of the following conditions are satisfied:*

1. *The set $\{\phi_i\}_{i=0}^{n-1}$ is linearly independent in U^*.*

2. *The set $\{\psi_i\}_{i=0}^{n-1}$ is linearly independent in V^*.*

3. *The families of linear functionals $\{\phi_i\}_{i=0}^{n-1}$ and $\{\psi_i\}_{i=0}^{n-1}$ commute over W.*

*For $0 \le i \le n-1$, let Φ_i and Ψ_i denote commuting extensions of $\phi_i \otimes I$ and $I \otimes \psi_i$ from $U \otimes V$ to W, respectively. Note that by assumption, $\Phi_i \Psi_j = \Psi_j \Phi_i$ for $0 \le i, j \le n-1$. Let $w_1, w_2 \in W$. We say that w_1 **interpolates** w_2 **strongly with respect to** Π_n if*

$$\Phi_i w_1 = \Phi_i w_2 \quad \text{and} \quad \Psi_i w_1 = \Psi_i w_2 \quad \text{for} \quad 0 \le i \le n-1.$$

*We say that w_1 **interpolates** w_2 **weakly with respect to** Π_n if*

$$\Phi_i \Psi_j w_1 = \Phi_i \Psi_j w_2 \quad \text{for} \quad 0 \le i, j \le n-1.$$

*We refer to these two kinds of interpolation over (U, V, W) as **strong** Π_n **interpolation** and **weak** Π_n **interpolation**, respectively.*

Let us now consider some of the consequences of this definition. Since $\Phi_i|U = \phi_i$ and $\Psi_i|V = \psi_i$ for $0 \le i \le n-1$, the linear independence of the linear functionals $\{\phi_i\}_{i=0}^{n-1}$ and $\{\psi_i\}_{i=0}^{n-1}$ automatically ensures the linear independence of the linear operators $\{\Phi_i\}_{i=0}^{n-1}$ and $\{\Psi_i\}_{i=0}^{n-1}$. This in turn eliminates unnecessary redundancy in the data $\{\Phi_i w\}_{i=0}^{n-1} \subset V$ and $\{\Psi_i w\}_{i=0}^{n-1} \subset U$ and $\{\Phi_i \Psi_j w\}_{i,j=0}^{n-1} \subset F$ which are generated by an arbitrary vector $w \in W$.

Note that strong Π_n interpolation always implies weak Π_n interpolation. In addition, both strong Π_n interpolation and weak Π_n interpolation are *equivalence relations* on the vector space W. If we define the subspace $\operatorname{strong}\Pi_n \subset W$ by

$$\operatorname{strong}\Pi_n := \bigcap_{i=0}^{n-1}(\ker\Phi_i \cap \ker\Psi_i),$$

the relation that w_1 interpolates w_2 strongly with respect to Π_n is equivalent to the condition $w_1 - w_2 \in \operatorname{strong}\Pi_n$. This means that the equivalence classes of the strong equivalence relation are precisely the elements of the quotient space $W/\operatorname{strong}\Pi_n$. Similarly, if we define the subspace $\operatorname{weak}\Pi_n \subset W$ by

$$\operatorname{weak}\Pi_n := \bigcap_{i,j=0}^{n-1}\ker(\Phi_i\Psi_j),$$

the relation that w_1 interpolates w_2 weakly with respect to Π_n is equivalent to the condition $w_1 - w_2 \in \operatorname{weak}\Pi_n$. The equivalence classes of the weak equivalence relation are precisely the elements of the quotient space $W/\operatorname{weak}\Pi_n$. Note that

$$\operatorname{strong}\Pi_n \subset \operatorname{weak}\Pi_n \subset W.$$

In the terminology of the previous definition, the tensor product $\Omega_{(\phi,\psi)}w \in U \otimes V$ interpolates the vector $w \in W_{(\phi,\psi)}$ both strongly and weakly with respect to the interpolation problem $(\phi;\psi) \in U^* \times V^*$ of order one over (U, V, W). The following concrete example will further illustrate the abstract formalisms of the previous definition. In addition, the ensuing discussion of this example will establish an important connection between the newly defined general interpolation problem and the previously studied problem of interpolation on a grid:

Example 10.9 (Grid Interpolation) *Let X and Y be arbitrary sets, and let F be a field. Define the function spaces $U := F^X$ and $V := F^Y$ and $W := F^{X\times Y}$, and recall that (U, V, W) is a linear configuration over F. Let n be a positive integer, and choose distinct $x_0, x_1, \ldots, x_{n-1} \in X$ and distinct $y_0, y_1, \ldots, y_{n-1} \in Y$. For $0 \le i \le n-1$, let $\phi_i := \varepsilon_{x_i} \in U^*$ and $\psi_i := \varepsilon^{y_i} \in V^*$ denote evaluation functionals, and let $\Phi_i := E_{x_i} \in \operatorname{Hom}_F(W, V)$ and $\Psi_i := E^{y_i} \in \operatorname{Hom}_F(W, U)$ denote the corresponding evaluation operators. Since the set $\{\phi_i\}_{i=0}^{n-1}$ is linearly independent in U^*, the set $\{\psi_i\}_{i=0}^{n-1}$ is linearly independent in V^*, and the families $\{\phi_i\}_{i=0}^{n-1}$ and $\{\psi_i\}_{i=0}^{n-1}$ commute over W, we conclude that*

$$\Pi_n := (\phi_0, \phi_1, \ldots, \phi_{n-1}; \psi_0, \psi_1, \ldots, \psi_{n-1}) \in (U^*)^n \times (V^*)^n$$

is an interpolation problem of order n over (U,V,W) by Definition 10.8.

Assuming the hypotheses of the previous example, and invoking the terminology of Section 5.2, we note that

$$\Gamma_n := (x_0, x_1, \ldots, x_{n-1}; y_0, y_1, \ldots, y_{n-1}) \in X^n \times Y^n$$

is a simple rectangular grid of order n in $X \times Y$. The abstract problem of *strong* Π_n interpolation corresponds to the concrete problem of interpolation on the $2n$ *lines* of the grid Γ_n, as defined in Sections 5.1 and 5.2. The abstract problem of *weak* Π_n interpolation corresponds to the concrete problem of interpolation on the n^2 *intersection points* of the grid Γ_n, which automatically occurs whenever we interpolate on the lines of the grid Γ_n.

Note that there is a standard method for solving the weak Π_n interpolation problem of the previous example using bivariate polynomials. There is also a standard method for solving the strong Π_n interpolation problem of the previous example using more general functions which are a hybrid of univariate polynomials and arbitrary univariate functions. The following example describes both of these standard interpolation techniques:

Example 10.10 (Polynomial Interpolation) *Assume the hypotheses of Example 10.9. In addition, assume that $F \subset \mathbb{C}$ is a subfield so that we can identify polynomials in the algebras $F[x]$ and $F[y]$ and $F[x,y]$ with functions in the algebras U and V and W. Note that since U and V are commutative rings with identity, we can form the algebra $V[x]$ of polynomials in x with coefficients which are functions of y, and the algebra $U[y]$ of polynomials in y with coefficients which are functions of x. It follows that*

$$V[x] = F[x] \otimes V \subset U \otimes V \quad \textit{and} \quad U[y] = U \otimes F[y] \subset U \otimes V.$$

For any $u \in U$, let $P_n u \in F[x]$ denote the unique polynomial of degree $n-1$ which interpolates u on the n distinct points $\{x_i\}_{i=0}^{n-1} \subset X$. Similarly, for any $v \in V$, let $Q_n v \in F[y]$ denote the unique polynomial of degree $n-1$ which interpolates v on the n distinct points $\{y_i\}_{i=0}^{n-1} \subset Y$. The interpolation operators P_n and Q_n are linear and idempotent, hence projections, and satisfy

$$P_n \in \mathrm{Hom}_F(U, F[x]) \quad \textit{and} \quad Q_n \in \mathrm{Hom}_F(V, F[y]).$$

Their parametric extensions $\tilde{P}_n$ *and* $\tilde{Q}_n$ *satisfy*

$$\tilde{P}_n \in \mathrm{Hom}_F(W, V[x]) \quad \textit{and} \quad \tilde{Q}_n \in \mathrm{Hom}_F(W, U[y]).$$

In addition, $\tilde{P}_n$ *and* $\tilde{Q}_n$ *commute, and their composition satisfies*

$$\tilde{P}_n\tilde{Q}_n = \tilde{Q}_n\tilde{P}_n \in \mathrm{Hom}_F(W, F[x,y]).$$

As an interesting aside, note that the restriction of the composition $\tilde{P}_n\tilde{Q}_n$ *to the tensor product space* $U \otimes V$ *satisfies*

$$(\tilde{P}_n\tilde{Q}_n)|(U \otimes V) = P_n \otimes Q_n : U \otimes V \to F[x] \otimes F[y] = F[x,y].$$

According to [CL00, Ch. 7], the polynomial $\tilde{P}_n\tilde{Q}_n f \in F[x,y]$ *interpolates the function* $f \in W$ *weakly with respect to* Π_n*, which means that* $\tilde{P}_n\tilde{Q}_n f$ *interpolates* f *on the intersection points of the grid* Γ_n*.*

Let us now define the subspace of ***hybrid polynomials in*** $U \otimes V$ *to be*

$$H := V[x] + U[y] + F[x,y] \subset U \otimes V.$$

Note that the elements of H *are generated by functions which are polynomial in at least one variable at a time. The* ***Boolean sum of*** $\tilde{P}_n$ ***and*** $\tilde{Q}_n$ *is the linear map* $\tilde{P}_n \oplus \tilde{Q}_n \in \mathrm{Hom}_F(W, H)$ *defined by*

$$\tilde{P}_n \oplus \tilde{Q}_n := \tilde{P}_n + \tilde{Q}_n - \tilde{P}_n\tilde{Q}_n.$$

According to [CL00, Ch. 8], the hybrid polynomial $(\tilde{P}_n \oplus \tilde{Q}_n)f \in H$ *interpolates the function* $f \in W$ *strongly with respect to* Π_n*, which means that* $(\tilde{P}_n \oplus \tilde{Q}_n)f$ *interpolates* f *on the lines of the grid* Γ_n*.*

In the notation of the previous example, the inclusion

$$F[x,y] \subset H \subset U \otimes V$$

tells us that bivariate polynomials and hybrid polynomials are both tensor products—the former being a special case of the latter. As we learned in the previous example, the smaller class of bivariate polynomials suffices to solve the weak Π_n interpolation problem, whereas the larger class of hybrid polynomials suffices to solve the strong Π_n interpolation problem.

Note that the nonpolynomial univariate functions which occur in the hybrid polynomial interpolant $(\tilde{P}_n \oplus \tilde{Q}_n)f \in H$ are actually the *cross-sections* of the original function $f \in W$ over the lines of the grid Γ_n. In Section 5.2, we used these *same* cross-sections to construct a *natural* tensor product which *also* solves the strong Π_n interpolation problem. This fact has the following important implication:

Remark 10.11 *Since we can solve the strong Π_n interpolation problem of Example 10.9 using nothing but cross-sections, the polynomial contributions to the hybrid polynomial solution of Example 10.10 are actually superfluous!*

What price do we pay for eliminating the polynomial contributions from the hybrid interpolant? The answer is that we must give up *linear* interpolation operators in exchange for *nonlinear* splitting operators; however, in doing so, we gain the *iterative* interpolation algorithms of the thesis. This iterative approach to tensor product interpolation offers significant advantages to constructive approximation theory, such as an efficient algorithm for iterative refinement and the automatic reduction of tensor products to binormal form. Since we clearly get the better end of this trade, we conclude that the traditional advantages of linearity are somewhat overrated in this particular context.

10.6 The Space of Natural Tensor Products

We will now define natural tensor products formally, and quickly develop their interpolation properties. We begin with the following original definition, which explains how an interpolation problem defines a *subspace of natural tensor products* within a vector space of tensor products:

Definition 10.12 *Let (U, V, W) be a linear configuration over a field F. Let n be a positive integer, and let*

$$\Pi_n := (\phi_0, \phi_1, \ldots, \phi_{n-1}; \psi_0, \psi_1, \ldots, \psi_{n-1}) \in (U^*)^n \times (V^*)^n$$

be an interpolation problem of order n over (U, V, W). For $0 \le i \le n-1$, let Φ_i and Ψ_i denote commuting extensions of $\phi_i \otimes I$ and $I \otimes \psi_i$ from $U \otimes V$ to W, respectively. Given a vector $w \in W$, define the following set of n^2 dyads in $U \otimes V$:

$$G(\Pi_n; w) := \{(\Psi_i \otimes \Phi_j)(w \otimes w)\}_{i,j=0}^{n-1} := \{(\Psi_i w) \otimes (\Phi_j w)\}_{i,j=0}^{n-1} \subset U \otimes V.$$

The ***space of natural tensor products generated by*** Π_n ***and*** w*, denoted by* $\mathrm{NTP}(\Pi_n; w)$*, is defined via*

$$\mathrm{NTP}(\Pi_n; w) := \mathrm{span}_F\, G(\Pi_n; w) \subset U \otimes V.$$

Equivalently, we could define the space $\mathrm{NTP}(\Pi_n; w)$ in the following way as the tensor product of finite-dimensional subspaces of U and V:

$$\mathrm{NTP}(\Pi_n; w) := \mathrm{span}_F\{\Psi_i w\}_{i=0}^{n-1} \otimes \mathrm{span}_F\{\Phi_i w\}_{i=0}^{n-1} \subset U \otimes V.$$

Since the space $\mathrm{NTP}(\Pi_n; w)$ is generated by the set $G(\Pi_n; w)$ of n^2 dyads, it follows immediately that

$$\dim_F \mathrm{NTP}(\Pi_n; w) \leq n^2 \quad \text{for all} \quad w \in W.$$

The above inequality becomes an equality for a particular vector $w \in W$ if and only if $\{\Psi_i w\}_{i=0}^{n-1} \subset U$ and $\{\Phi_i w\}_{i=0}^{n-1} \subset V$ are both linearly independent sets. In that case, $G(\Pi_n; w)$ is a basis of the space $\mathrm{NTP}(\Pi_n; w)$ by Hamel Basis Theorem 4.14.

Note that every element of $\mathrm{NTP}(\Pi_n; w)$ can be fully described in terms of the n^2 generators in $G(\Pi_n; w)$ by specifying n^2 elements of F. It therefore seems reasonable to develop a representation for the elements of $\mathrm{NTP}(\Pi_n; w)$ using $n \times n$ matrices over F. To that end, we will find it very helpful to define some vectors whose elements are linear functionals and some vectors whose elements are linear operators.

As a convenience, let us define the following n-dimensional vectors of linear functionals:

$$\boldsymbol{\phi}_n := [\phi_i]_{i=0}^{n-1} \in \mathrm{Col}_n(U^*) \quad \text{and} \quad \boldsymbol{\psi}_n := [\psi_j]_{j=0}^{n-1} \in \mathrm{Row}_n(V^*).$$

From now on, we will freely identify the column vector $\boldsymbol{\phi}_n$ with the n-tuple $(\phi_0, \phi_1, \ldots, \phi_{n-1})$ and the row vector $\boldsymbol{\psi}_n$ with the n-tuple $(\psi_0, \psi_1, \ldots, \psi_{n-1})$. These identifications will allow us to write the original interpolation problem as

$$\Pi_n := (\boldsymbol{\phi}_n; \boldsymbol{\psi}_n) \in \mathrm{Col}_n(U^*) \times \mathrm{Row}_n(V^*).$$

This new notation also allows us to denote the space $\mathrm{NTP}(\Pi_n; w)$ as $\mathrm{NTP}(\boldsymbol{\phi}_n; \boldsymbol{\psi}_n; w)$ and the set of generators $G(\Pi_n; w)$ as $G(\boldsymbol{\phi}_n; \boldsymbol{\psi}_n; w)$.

Let us now define the following n-dimensional vectors of corresponding linear operators:

$$\boldsymbol{\Phi}_n := [\Phi_i]_{i=0}^{n-1} \in \mathrm{Col}_n(\mathrm{Hom}_F(W, V)) \quad \text{and} \quad \boldsymbol{\Psi}_n := [\Psi_j]_{j=0}^{n-1} \in \mathrm{Row}_n(\mathrm{Hom}_F(W, U)).$$

From now on, it will be understood that if $\Pi_n := (\boldsymbol{\phi}_n; \boldsymbol{\psi}_n)$ defines an interpolation problem of order n over (U, V, W), then $\boldsymbol{\Phi}_n$ and $\boldsymbol{\Psi}_n$ will denote the n-dimensional vectors of linear operators corresponding to the n-dimensional vectors $\boldsymbol{\phi}_n$ and $\boldsymbol{\psi}_n$ of linear functionals.

We define the *noncommuting* product $\boldsymbol{\Phi}_n \boldsymbol{\Psi}_n$ by ordinary matrix multiplication, where the product of the matrix elements is understood to be the *composition of linear operators*. This yields the following matrix of linear functionals:

$$\boldsymbol{\Phi}_n \boldsymbol{\Psi}_n := [\Phi_i \Psi_j]_{i,j=0}^{n-1} \in \mathrm{Mat}_n(W^*).$$

We define the actions of $\boldsymbol{\Phi}_n$ and $\boldsymbol{\Psi}_n$ and $\boldsymbol{\Phi}_n \boldsymbol{\Psi}_n$ on a vector $w \in W$ in a componentwise fashion as follows:

$$\begin{array}{lcll}
\boldsymbol{\Phi}_n w & := & [\Phi_i w]_{i=0}^{n-1} & \in \ \mathrm{Col}_n(V) \\
\boldsymbol{\Psi}_n w & := & [\Psi_j w]_{j=0}^{n-1} & \in \ \mathrm{Row}_n(U) \\
\boldsymbol{\Phi}_n \boldsymbol{\Psi}_n w & := & [\Phi_i \Psi_j w]_{i,j=0}^{n-1} & \in \ \mathrm{Mat}_n(F).
\end{array}$$

Given any coefficient matrix $C_n := [c_{ij}]_{i,j=0}^{n-1} \in \mathrm{Mat}_n(F)$, we define the natural tensor product

$$(\boldsymbol{\Psi}_n w)\, C_n\, (\boldsymbol{\Phi}_n w) \in \mathrm{NTP}(\boldsymbol{\phi}_n; \boldsymbol{\psi}_n; w)$$

by ordinary vector-matrix-vector multiplication, where the product of the elements is understood to be the *tensor product*. Using the identification $U \otimes F \otimes V = U \otimes V$ yields the explicit formula

$$(\boldsymbol{\Psi}_n w)\, C_n\, (\boldsymbol{\Phi}_n w) := \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} c_{ij} \cdot (\Psi_i \otimes \Phi_j)(w \otimes w) \in \mathrm{NTP}(\boldsymbol{\phi}_n; \boldsymbol{\psi}_n; w).$$

Since the space $\mathrm{NTP}(\boldsymbol{\phi}_n; \boldsymbol{\psi}_n; w)$ is generated by the set $G(\boldsymbol{\phi}_n; \boldsymbol{\psi}_n; w)$, it is clear that every element of $\mathrm{NTP}(\boldsymbol{\phi}_n; \boldsymbol{\psi}_n; w)$ can be represented as $(\boldsymbol{\Psi}_n w)\, C_n\, (\boldsymbol{\Phi}_n w)$ for some coefficient matrix $C_n \in \mathrm{Mat}_n(F)$.

We will now use our matrix representation for natural tensor products to state the following original theorem. The proof of this theorem is straightforward and is omitted for the sake of brevity.

Theorem 10.13 *Let (U, V, W) be a linear configuration over a field F. Let n be a positive integer, and let*

$$\Pi_n := (\boldsymbol{\phi}_n; \boldsymbol{\psi}_n) \in \mathrm{Col}_n(U^*) \times \mathrm{Row}_n(V^*)$$

be an interpolation problem of order n over (U, V, W). Let $w \in W$. If the matrix $\boldsymbol{\Phi}_n \boldsymbol{\Psi}_n w \in$

$\mathrm{Mat}_n(F)$ *is invertible, then* $G(\boldsymbol{\phi}_n; \boldsymbol{\psi}_n; w)$ *is a basis of the space* $\mathrm{NTP}(\boldsymbol{\phi}_n; \boldsymbol{\psi}_n; w)$*, and every element of* $\mathrm{NTP}(\boldsymbol{\phi}_n; \boldsymbol{\psi}_n; w)$ *has a unique matrix representation* $(\boldsymbol{\Psi}_n w)\, C_n\, (\boldsymbol{\Phi}_n w)$ *for some coefficient matrix* $C_n \in \mathrm{Mat}_n(F)$*. In addition, there is a unique natural tensor product in* $\mathrm{NTP}(\boldsymbol{\phi}_n; \boldsymbol{\psi}_n; w)$ *which interpolates the vector* w *strongly with respect to* $\Pi_n := (\boldsymbol{\phi}_n; \boldsymbol{\psi}_n)$*, and its coefficient matrix is given by*

$$C_n = (\boldsymbol{\Phi}_n \boldsymbol{\Psi}_n w)^{-1} \in \mathrm{Mat}_n(F).$$

The previous theorem leads immediately to the following fundamental definition:

Definition 10.14 *Assume the general hypotheses of Theorem 10.13. If the matrix* $\boldsymbol{\Phi}_n \boldsymbol{\Psi}_n w \in \mathrm{Mat}_n(F)$ *is invertible, we call*

$$\sigma_n := (\boldsymbol{\Psi}_n w)(\boldsymbol{\Phi}_n \boldsymbol{\Psi}_n w)^{-1}(\boldsymbol{\Phi}_n w) \in \mathrm{NTP}(\boldsymbol{\phi}_n; \boldsymbol{\psi}_n; w) \subset U \otimes V \subset W$$

the ***natural tensor product interpolant for*** w ***with respect to*** $\Pi_n := (\boldsymbol{\phi}_n; \boldsymbol{\psi}_n)$*. We call the vector*

$$\rho_n := w - \sigma_n \in W$$

the ***remainder associated with*** σ_n*.*

Let us now define two $(n+1)$-dimensional vectors $\widetilde{\boldsymbol{\Phi}}_n$ and $\widetilde{\boldsymbol{\Psi}}_n$ of linear operators by augmenting the n-dimensional vectors $\boldsymbol{\Phi}_n$ and $\boldsymbol{\Psi}_n$ of linear operators with the identity operator I in the following way:

$$\begin{aligned} \widetilde{\boldsymbol{\Phi}}_n &:= \begin{bmatrix} \boldsymbol{\Phi}_n \\ I \end{bmatrix} &\in\ & \mathrm{Col}_{n+1}(\mathrm{Hom}_F(W,W)) \\ \widetilde{\boldsymbol{\Psi}}_n &:= \begin{bmatrix} \boldsymbol{\Psi}_n & I \end{bmatrix} &\in\ & \mathrm{Row}_{n+1}(\mathrm{Hom}_F(W,W)). \end{aligned}$$

The matrix product $\widetilde{\boldsymbol{\Phi}}_n \widetilde{\boldsymbol{\Psi}}_n$ can be written as a partitioned matrix as follows:

$$\widetilde{\boldsymbol{\Phi}}_n \widetilde{\boldsymbol{\Psi}}_n = \begin{bmatrix} \boldsymbol{\Phi}_n \boldsymbol{\Psi}_n & \boldsymbol{\Phi}_n \\ \boldsymbol{\Psi}_n & I \end{bmatrix} \in \mathrm{Mat}_{n+1}(\mathrm{Hom}_F(W,W)).$$

When we apply the matrix product $\widetilde{\mathbf{\Phi}}_n\widetilde{\mathbf{\Psi}}_n$ to a vector $w \in W$ in a componentwise fashion, we obtain the following partitioned matrix:

$$\widetilde{\mathbf{\Phi}}_n\widetilde{\mathbf{\Psi}}_n w = \begin{bmatrix} \mathbf{\Phi}_n\mathbf{\Psi}_n w & \mathbf{\Phi}_n w \\ \mathbf{\Psi}_n w & w \end{bmatrix} \in \mathrm{Mat}_{n+1}(W).$$

We can calculate the determinant $\det \widetilde{\mathbf{\Phi}}_n\widetilde{\mathbf{\Psi}}_n w$ via expansion by cofactors along either the last row or the last column of the matrix $\widetilde{\mathbf{\Phi}}_n\widetilde{\mathbf{\Psi}}_n w$, where the product of the elements of $\widetilde{\mathbf{\Phi}}_n\widetilde{\mathbf{\Psi}}_n w$ is understood to be the *tensor product*.[4] With this interpretation, we obtain a well-defined vector $\det \widetilde{\mathbf{\Phi}}_n\widetilde{\mathbf{\Psi}}_n w \in W$.

We are now ready to use our earlier results on partitioned matrices to prove a standard result which expresses the remainder ρ_n associated with the natural tensor product interpolant σ_n as the ratio of two determinants:

Theorem 10.15 (Determinant Remainder) *Assume the general hypotheses of Theorem 10.13, and assume that the matrix $\mathbf{\Phi}_n\mathbf{\Psi}_n w \in \mathrm{Mat}_n(F)$ is invertible. If σ_n denotes the the natural tensor product interpolant for w with respect to $\Pi_n := (\boldsymbol{\phi}_n; \boldsymbol{\psi}_n)$, then the associated remainder ρ_n can be written as the ratio of two determinants in the following way:*

$$\rho_n = \frac{\det \widetilde{\mathbf{\Phi}}_n\widetilde{\mathbf{\Psi}}_n w}{\det \mathbf{\Phi}_n\mathbf{\Psi}_n w}.$$

Proof. Since $\mathbf{\Phi}_n\mathbf{\Psi}_n w$ is an invertible submatrix of the partitioned matrix $\widetilde{\mathbf{\Phi}}_n\widetilde{\mathbf{\Psi}}_n w$, the Schur complement of $\mathbf{\Phi}_n\mathbf{\Psi}_n w$ in $\widetilde{\mathbf{\Phi}}_n\widetilde{\mathbf{\Psi}}_n w$ is well-defined and is given by

$$\overline{\mathbf{\Phi}_n\mathbf{\Psi}_n w} := w - (\mathbf{\Psi}_n w)(\mathbf{\Phi}_n\mathbf{\Psi}_n w)^{-1}(\mathbf{\Phi}_n w) = w - \sigma_n =: \rho_n.$$

Since ρ_n is an element of $\mathrm{Mat}_1(W) = W$, it follows that

$$\det \overline{\mathbf{\Phi}_n\mathbf{\Psi}_n w} = \det \rho_n = \rho_n.$$

The Schur determinant formula for the partitioned matrix $\widetilde{\mathbf{\Phi}}_n\widetilde{\mathbf{\Psi}}_n w$ implies that

$$\det \widetilde{\mathbf{\Phi}}_n\widetilde{\mathbf{\Psi}}_n w = \det \mathbf{\Phi}_n\mathbf{\Psi}_n w \cdot \det \overline{\mathbf{\Phi}_n\mathbf{\Psi}_n w} = \det \mathbf{\Phi}_n\mathbf{\Psi}_n w \cdot \rho_n.$$

[4] Recall that a similar device is used to define the cross product of two vectors in three-dimensional Euclidean space using a special determinant which contains a row of basis vectors. In that case, the product of a cofactor and a basis vector is understood to be the *scalar product*.

Since $\det \mathbf{\Phi}_n \mathbf{\Psi}_n w$ is a nonzero scalar, we can solve the previous equation for ρ_n to obtain the desired result. ■

The previous theorem has the following important implication:

Corollary 10.16 (Invariance) *The natural tensor product interpolant σ_n is invariant under permutations of the linear functionals $\phi_0, \phi_1, \ldots, \phi_{n-1}$ as well as permutations of the linear functionals $\psi_0, \psi_1, \ldots, \psi_{n-1}$.*

Proof. Transposing the linear functionals ϕ_i and ϕ_j interchanges rows i and j in both the numerator and denominator of

$$\rho_n = \frac{\det \widetilde{\mathbf{\Phi}}_n \widetilde{\mathbf{\Psi}}_n w}{\det \mathbf{\Phi}_n \mathbf{\Psi}_n w}.$$

This changes the sign of both determinants, and the two minus signs cancel, leaving ρ_n unchanged. Similarly, transposing the linear functionals ψ_i and ψ_j interchanges columns i and j, which also leaves ρ_n unchanged. Since w is completely independent of these linear functionals, the difference $\sigma_n = w - \rho_n$ is also invariant under such transpositions. Since every permutation is the product of transpositions, σ_n is invariant under arbitrary permutations as well. ■

10.7 Matrix Augmentation and Iterative Interpolation

The most costly step in constructing the natural tensor product interplant σ_n is inverting the $n \times n$ matrix $\mathbf{\Phi}_n \mathbf{\Psi}_n w$. If our ultimate reason for constructing σ_n is to generate a good approximation for w, we may need to carry out this interpolation process for higher and higher values of n until the desired level of accuracy is achieved. Clearly, the cost of inverting a sequence of larger and larger matrices by any direct method will be prohibitive in many applications for which constructing the approximation σ_n to w is merely a preliminary step in the solution of some larger problem. What are we to do?

Remark 10.17 *The iterative interpolation algorithms of the thesis offer an elegant and highly efficient way to overcome precisely this difficulty!*

The key is to augment the original interpolation problem $\Pi_n := (\boldsymbol{\phi}_n; \boldsymbol{\psi}_n)$ by retaining all the original linear functions and simply adding new linear functionals. This in turn will augment the $n \times n$ matrix $\mathbf{\Phi}_n \mathbf{\Psi}_n w$ to form a larger matrix which has $\mathbf{\Phi}_n \mathbf{\Psi}_n w$ as a submatrix. By exploiting the block structure of the augmented matrix, we can calculate its inverse at a small fraction of the cost of direct matrix inversion—simply by reusing the already-calculated inverse of the submatrix $\mathbf{\Phi}_n\mathbf{\Psi}_n w$. The following original theorem explains exactly how to do this—using the abstract splitting operator!

Theorem 10.18 (Augmentation) *Assume the general hypotheses of Theorem 10.13, and assume that the matrix $\mathbf{\Phi}_n\mathbf{\Psi}_n w \in \mathrm{Mat}_n(F)$ is invertible. Let σ_n denote the the natural tensor product interpolant for w with respect to $\Pi_n := (\boldsymbol{\phi}_n; \boldsymbol{\psi}_n)$, and let ρ_n denote the associated remainder. Choose two new linear functionals $\phi_n \in U^*$ and $\psi_n \in V^*$ such that*

$$\Pi_{n+1} := (\phi_0, \phi_1, \ldots, \phi_{n-1}, \phi_n; \psi_0, \psi_1, \ldots, \psi_{n-1}, \psi_n) \in (U^*)^{n+1} \times (V^*)^{n+1}$$

is an interpolation problem of order $n+1$ over (U, V, W). If the scalar $\Phi_n\Psi_n\rho_n \in F$ is invertible, it follows that the matrix $\mathbf{\Phi}_{n+1}\mathbf{\Psi}_{n+1}w \in \mathrm{Mat}_{n+1}(F)$ is also invertible. The natural tensor product interpolant σ_{n+1} for w with respect to $\Pi_{n+1} := (\boldsymbol{\phi}_{n+1}; \boldsymbol{\psi}_{n+1})$ is therefore well-defined and satisfies

$$\sigma_{n+1} = \sigma_n + \Omega_{(\phi_n, \psi_n)}\rho_n.$$

Proof. Since $\Phi_n\Psi_n\rho_n \neq 0$ by hypothesis, it follows that $\rho_n \in W_{(\phi_n,\psi_n)}$, and thus $\Omega_{(\phi_n,\psi_n)}\rho_n \in U \otimes V$ is well-defined. The natural tensor product interpolant σ_{n+1} is well-defined if and only if the following matrix is invertible:

$$\mathbf{\Phi}_{n+1}\mathbf{\Psi}_{n+1}w = \begin{bmatrix} \mathbf{\Phi}_n \\ \Phi_n \end{bmatrix} \begin{bmatrix} \mathbf{\Psi}_n & \Psi_n \end{bmatrix} w = \begin{bmatrix} \mathbf{\Phi}_n\mathbf{\Psi}_n w & \mathbf{\Phi}_n\Psi_n w \\ \Phi_n\mathbf{\Psi}_n w & \Phi_n\Psi_n w \end{bmatrix} \in \mathrm{Mat}_{n+1}(F).$$

Since $\mathbf{\Phi}_n\mathbf{\Psi}_n w$ is an invertible submatrix of the partitioned matrix $\mathbf{\Phi}_{n+1}\mathbf{\Psi}_{n+1}w$, the Schur complement of $\mathbf{\Phi}_n\mathbf{\Psi}_n w$ in $\mathbf{\Phi}_{n+1}\mathbf{\Psi}_{n+1}w$ is well-defined and is given by

$$\begin{aligned} \overline{\mathbf{\Phi}_n\mathbf{\Psi}_n w} &:= \Phi_n\Psi_n w - (\Phi_n\mathbf{\Psi}_n w)(\mathbf{\Phi}_n\mathbf{\Psi}_n w)^{-1}(\mathbf{\Phi}_n\Psi_n w) \\ &= \Phi_n\Psi_n w - \Phi_n\Psi_n[(\mathbf{\Psi}_n w)(\mathbf{\Phi}_n\mathbf{\Psi}_n w)^{-1}(\mathbf{\Phi}_n w)] \\ &= \Phi_n\Psi_n w - \Phi_n\Psi_n\sigma_n = \Phi_n\Psi_n(w - \sigma_n) = \Phi_n\Psi_n\rho_n \neq 0. \end{aligned}$$

The above derivation is justified since Φ_i and Ψ_j commute for $0 \leq i, j \leq n-1$.

Since both $\mathbf{\Phi}_n\mathbf{\Psi}_n w$ and $\overline{\mathbf{\Phi}_n\mathbf{\Psi}_n w}$ are invertible, the partitioned matrix $\mathbf{\Phi}_{n+1}\mathbf{\Psi}_{n+1}w$ is also invertible, which means that σ_{n+1} is well-defined. For convenience, assign

$$\begin{aligned}
\mathbf{v}_n &:= \mathbf{\Phi}_n w &\in \mathrm{Col}_n(V)\\
\mathbf{u}_n &:= \mathbf{\Psi}_n w &\in \mathrm{Row}_n(U)\\
M_n &:= \mathbf{\Phi}_n\mathbf{\Psi}_n w &\in \mathrm{Mat}_n(F)\\
\overline{M_n} &:= \Phi_n\Psi_n\rho_n &\in F.
\end{aligned}$$

Note the following formula, which we will use frequently in the remainder of the proof:

$$\sigma_n = \mathbf{u}_n M_n^{-1}\mathbf{v}_n.$$

We can rewrite the partitioned matrix $\mathbf{\Phi}_{n+1}\mathbf{\Psi}_{n+1}w$ as

$$\mathbf{\Phi}_{n+1}\mathbf{\Psi}_{n+1}w = \begin{bmatrix} M_n & \Psi_n\mathbf{v}_n \\ \Phi_n\mathbf{u}_n & \Phi_n\Psi_n w \end{bmatrix} \in \mathrm{Mat}_{n+1}(F).$$

The formula for the inverse of a partitioned matrix yields

$$(\mathbf{\Phi}_{n+1}\mathbf{\Psi}_{n+1}w)^{-1} = \begin{bmatrix} M_n^{-1} + \dfrac{M_n^{-1}(\Psi_n\mathbf{v}_n)(\Phi_n\mathbf{u}_n)M_n^{-1}}{\overline{M_n}} & -\dfrac{M_n^{-1}(\Psi_n\mathbf{v}_n)}{\overline{M_n}} \\[2ex] -\dfrac{(\Phi_n\mathbf{u}_n)M_n^{-1}}{\overline{M_n}} & \dfrac{1}{\overline{\overline{M_n}}} \end{bmatrix}.$$

From this, we obtain the natural tensor product interpolant

$$
\begin{aligned}
\sigma_{n+1} \quad &:= \quad (\boldsymbol{\Psi}_{n+1} w)(\boldsymbol{\Phi}_{n+1}\boldsymbol{\Psi}_{n+1} w)^{-1}(\boldsymbol{\Phi}_{n+1} w) = \begin{bmatrix} \boldsymbol{\Psi}_n w & \Psi_n w \end{bmatrix} (\boldsymbol{\Phi}_{n+1}\boldsymbol{\Psi}_{n+1} w)^{-1} \begin{bmatrix} \boldsymbol{\Phi}_n w \\ \Phi_n w \end{bmatrix} \\
&= \quad \begin{bmatrix} \mathbf{u}_n & \Psi_n w \end{bmatrix} \begin{bmatrix} M_n^{-1} + \dfrac{M_n^{-1}(\Psi_n \mathbf{v}_n)(\Phi_n \mathbf{u}_n) M_n^{-1}}{\overline{M}_n} & -\dfrac{M_n^{-1}(\Psi_n \mathbf{v}_n)}{\overline{M}_n} \\[2ex] -\dfrac{(\Phi_n \mathbf{u}_n) M_n^{-1}}{\overline{M}_n} & \dfrac{1}{\overline{M}_n} \end{bmatrix} \begin{bmatrix} \mathbf{v}_n \\ \Phi_n w \end{bmatrix} \\
&= \quad \begin{bmatrix} \mathbf{u}_n & \Psi_n w \end{bmatrix} \begin{bmatrix} M_n^{-1}\mathbf{v}_n + \dfrac{M_n^{-1}(\Psi_n \mathbf{v}_n)(\Phi_n \mathbf{u}_n) M_n^{-1}\mathbf{v}_n}{\overline{M}_n} - \dfrac{M_n^{-1}(\Psi_n \mathbf{v}_n)(\Phi_n w)}{\overline{M}_n} \\[2ex] -\dfrac{(\Phi_n \mathbf{u}_n) M_n^{-1}\mathbf{v}_n}{\overline{M}_n} + \dfrac{\Phi_n w}{\overline{M}_n} \end{bmatrix} \\
&= \quad \mathbf{u}_n M_n^{-1}\mathbf{v}_n + \frac{\mathbf{u}_n M_n^{-1}(\Psi_n \mathbf{v}_n)(\Phi_n \mathbf{u}_n) M_n^{-1}\mathbf{v}_n}{\overline{M}_n} - \frac{\mathbf{u}_n M_n^{-1}(\Psi_n \mathbf{v}_n)(\Phi_n w)}{\overline{M}_n} \\
&\qquad -\frac{(\Psi_n w)(\Phi_n \mathbf{u}_n) M_n^{-1}\mathbf{v}_n}{\overline{M}_n} + \frac{(\Psi_n w)(\Phi_n w)}{\overline{M}_n} \\
&= \quad \sigma_n + \frac{(\Psi_n \sigma_n) \otimes (\Phi_n \sigma_n) - (\Psi_n \sigma_n) \otimes (\Phi_n w) - (\Psi_n w) \otimes (\Phi_n \sigma_n) + (\Psi_n w) \otimes (\Phi_n w)}{\overline{M}_n} \\
&= \quad \sigma_n + \frac{(\Psi_n \sigma_n) \otimes \Phi_n(\sigma_n - w) - (\Psi_n w) \otimes \Phi_n(\sigma_n - w)}{\overline{M}_n} \\
&= \quad \sigma_n + \frac{\Psi_n(\sigma_n - w) \otimes \Phi_n(\sigma_n - w)}{\overline{M}_n} = \sigma_n + \frac{\Psi_n(w - \sigma_n) \otimes \Phi_n(w - \sigma_n)}{\overline{M}_n} \\
&= \quad \sigma_n + \frac{\Psi_n(\rho_n) \otimes \Phi_n(\rho_n)}{\overline{M}_n} = \sigma_n + \frac{(\Psi_n \otimes \Phi_n)(\rho_n \otimes \rho_n)}{(\Phi_n \Psi_n)(\rho_n)} = \sigma_n + \Omega_{(\phi_n,\psi_n)}\rho_n.
\end{aligned}
$$

This completes the proof. ■

With Augmentation Theorem 10.18 at our disposal, we are finally in a position to establish the connection between the traditional closed-form approach to natural tensor product interpolation and the new iterative approach set forth in the thesis. The following original theorem makes this connection explicit:

Theorem 10.19 (Iterative Interpolation) *Assume the general hypotheses of Theorem 10.13. Define $s_0 := 0$ and $r_0 := w$, and iterate the following for $i = 0, 1, \ldots, n-1$:*

$$s_{i+1} := s_i + \Omega_{(\phi_i,\psi_i)} r_i \quad \textit{and} \quad r_{i+1} := w - s_{i+1}.$$

In this iteration, we have implicitly assumed that

$$\Phi_i \Psi_i r_i \neq 0 \quad \textit{for} \quad 0 \leq i \leq n-1.$$

It follows from these assumptions that the matrix $\mathbf{\Phi}_n \mathbf{\Psi}_n w \in \mathrm{Mat}_n(F)$ is invertible, that the natural tensor product interpolant σ_n for w with respect to $\Pi_n := (\boldsymbol{\phi}_n; \boldsymbol{\psi}_n)$ is well-defined, and that

$$s_n = \sigma_n \quad \textit{and} \quad r_n = \rho_n.$$

Proof. Note that if we can show $s_n = \sigma_n$, it will always follow that $r_n = \rho_n$ since by definition

$$r_n := w - s_n = w - \sigma_n =: \rho_n.$$

The proof is by induction on n. Let P_n denote the logical proposition asserted by the theorem. To show that P_1 holds, we assume that $\Phi_0 \Psi_0 w \neq 0$ and calculate

$$\begin{aligned} s_1 \quad &:= \quad \Omega_{(\phi_0,\psi_0)} w = \frac{(\Psi_0 \otimes \Phi_0)(w \otimes w)}{(\Phi_0 \Psi_0)(w)} = \frac{(\Psi_0 w) \otimes (\Phi_0 w)}{\Phi_0 \Psi_0 w} \\ &= \quad (\Psi_0 w) \otimes (\Phi_0 \Psi_0 w)^{-1} \otimes (\Phi_0 w) = (\mathbf{\Psi}_1 w)(\mathbf{\Phi}_1 \mathbf{\Psi}_1 w)^{-1} (\mathbf{\Phi}_1 w) =: \sigma_1. \end{aligned}$$

It is implicit in this derivation that the matrix $\mathbf{\Phi}_1 \mathbf{\Psi}_1 w$ is invertible and that σ_1 is well-defined. This shows that proposition P_1 holds.

Now assume that proposition P_n holds for an arbitrary positive integer n. We must show that proposition P_{n+1} follows. Since proposition P_n implies that the matrix $\mathbf{\Phi}_n \mathbf{\Psi}_n w \in \mathrm{Mat}_n(F)$ is invertible and that $r_n = \rho_n$, the hypothesis $\Phi_n \Psi_n r_n \neq 0$ further implies that $\Phi_n \Psi_n \rho_n \neq 0$. We conclude by Augmentation Theorem 10.18 that the matrix $\mathbf{\Phi}_{n+1} \mathbf{\Psi}_{n+1} w \in \mathrm{Mat}_{n+1}(F)$ is invertible and that the natural tensor product interpolant σ_{n+1} for w with respect to $\Pi_{n+1} := (\boldsymbol{\phi}_{n+1}; \boldsymbol{\psi}_{n+1})$ is well-defined and satisfies

$$\sigma_{n+1} = \sigma_n + \Omega_{(\phi_n,\psi_n)} \rho_n.$$

Since $s_n = \sigma_n$ and $r_n = \rho_n$ by Proposition P_n, it follows from the definition of s_{n+1} that

$$s_{n+1} := s_n + \Omega_{(\phi_n,\psi_n)} r_n = \sigma_n + \Omega_{(\phi_n,\psi_n)} \rho_n = \sigma_{n+1},$$

which shows that proposition P_{n+1} holds.

We have shown that proposition P_1 holds, and that proposition P_{n+1} follows from proposition P_n for each positive integer n. By induction on n, proposition P_n holds for all positive integers n. ■

The previous theorem has the following immediate corollary:

Corollary 10.20 *Iterative Interpolation Theorem 10.19 implies that the matrix* $\boldsymbol{\Phi}_i \boldsymbol{\Psi}_i w \in \mathrm{Mat}_i(F)$ *is invertible, that* σ_i *is well-defined, and that* $s_i = \sigma_i$ *and* $r_i = \rho_i$ *for* $1 \leq i \leq n$.

The previous corollary further implies that we can perform Gaussian elimination on the matrix $\boldsymbol{\Phi}_n \boldsymbol{\Psi}_n w \in \mathrm{Mat}_n(F)$ without encountering an zero pivots. In fact, the pivots are given explicitly by the formula

$$\Phi_i \Psi_i r_i = \frac{\det \boldsymbol{\Phi}_{i+1} \boldsymbol{\Psi}_{i+1} w}{\det \boldsymbol{\Phi}_i \boldsymbol{\Psi}_i w} \neq 0 \quad \text{for} \quad 0 \leq i \leq n-1,$$

where $\det \boldsymbol{\Phi}_0 \boldsymbol{\Psi}_0 w = 1$ by convention. Conversely, if we can perform Gaussian elimination on the invertible matrix $\boldsymbol{\Phi}_n \boldsymbol{\Psi}_n w$ without encountering an zero pivots, then the iteration in Iterative Interpolation Theorem 10.19 can be successfully completed.

The following original definition is motivated by the previous theorem:

Definition 10.21 *Assume the general hypotheses of Theorem 10.13. In the notation of Iterative Interpolation Theorem 10.19, we call the tensor product series*

$$s_n := \sum_{i=0}^{n-1} \Omega_{(\phi_i,\psi_i)} r_i = \sum_{i=0}^{n-1} \frac{(\Psi_i r_i) \otimes (\Phi_i r_i)}{\Phi_i \Psi_i r_i}$$

the ***Geddes series for*** w ***generated by*** $\Pi_n := (\boldsymbol{\phi}_n; \boldsymbol{\psi}_n)$.

Note that we can also extend the previous definition to include the case $n = \infty$ by treating the resulting series as a formal series. Let us now rewrite Iterative Interpolation Theorem 10.19 in terms of the matrix form of the natural tensor product interpolant and the above Geddes series as follows:

$$\boxed{(\boldsymbol{\Psi}_n w)(\boldsymbol{\Phi}_n \boldsymbol{\Psi}_n w)^{-1}(\boldsymbol{\Phi}_n w) = \sum_{i=0}^{n-1} \frac{(\Psi_i r_i) \otimes (\Phi_i r_i)}{\Phi_i \Psi_i r_i}.}$$

This single equation captures the very essence of the thesis by distilling the central contribution of the thesis down to a one-line result! In this fundamental equation, we see the clear contrast between the traditional, closed-form, matrix-algebra approach to natural tensor product interpolation on the left-hand side and the author's new, iterative, series approach to natural tensor product interpolation on the right-hand side.

This contrast is best described by invoking the Lagrange and Newton paradigms for interpolation: The left-hand side uses a *Lagrange paradigm* for natural tensor product interpolation, which requires us to recalculate the entire matrix inverse every time we increase the order of the interpolation problem. In contrast, the right-hand side uses a *Newton paradigm* for natural tensor product interpolation, which allows us to reuse the results of a previous calculation by simply adding a new term to the existing series expansion. We can therefore summarize the central contribution of the thesis by stating that the Geddes series constitutes a Newton paradigm for natural tensor product interpolation, and thus complements the Lagrange paradigm for natural tensor product interpolation which is found in the mathematical literature.

In this chapter, we developed the theory of natural tensor product interpolation at a high level of abstraction using abstract linear functionals on abstract vector spaces. In the next chapter, we will develop a new class of generalized orthogonal series which is capable of accommodating all of the series expansions used to solve interpolation problems throughout this thesis.

Chapter 11

Generalized Orthogonal Series Expansions

Now that we have developed Geddes series expansions in the abstract setting of linear configurations of vector spaces, we will consider the sense in which Geddes series constitute generalized orthogonal series. We will also develop an analogous interpretation for dual asymptotic expansions. The key in both cases is to develop an appropriate generalization of classical inner product spaces.

Although generalized inner product spaces are by no means new, all prior studies with which the author is familiar adopt a heavily topological approach which is usually based on normed linear spaces over the real or complex numbers. In sharp contrast, the author has developed a purely algebraic approach based on abstract vector spaces over an arbitrary field. This approach was designed to meet the diverse needs of applications taken from all three traditional branches of mathematics—geometry, analysis, and algebra. Indeed, the author's algebraic axioms for generalized inner product spaces have evolved over the course of nearly two decades of mathematical research in these areas. This slow, patient, application-driven evolution has distilled the essence of these generalized inner product spaces down to two simple axioms whose power and applicability should not be underestimated. In this chapter of the thesis, we will develop these axioms and describe their applications to our exact series solutions to natural tensor product interpolation problems.

11.1 New Axioms for Generalized Inner Product Spaces

In this section, we will develop the author's original axioms for generalized inner product spaces. We will show that every classical inner product space over the real numbers is also an inner product space in this new generalized sense. We will also introduce a class of examples of generalized inner product spaces which are *not* classical inner product spaces—namely, the asymptotic inner product spaces.

A *classical* inner product on a vector space V over the field $\mathbb{R}$ is a positive-definite, symmetric, bilinear functional defined on *all* of $V \times V$. In contrast, a *generalized* inner product on a vector space V over an arbitrary field F is a definite, unilinear functional defined on a suitable *subset* of $V \times V$.

We will now describe what makes a subset "suitable" to be the domain of a generalized inner product. Recall that every subset $S \subset V \times V$ can be regarded as a relation on V. The set of all first coordinates in S is called the domain of S and is denoted by $\operatorname{dom} S$. Given any $u \in \operatorname{dom} S$, we define the image of u under S to be the set

$$S_u := \{v \in V : (u, v) \in S\}.$$

By construction, $S_u \subset V$.

Let $g : S \to F$ be an arbitrary bivariate functional defined on the relation $S \subset V \times V$. Given any $u \in \operatorname{dom} S$, we can define a univariate functional

$$g_u : S_u \to F \quad \text{by} \quad g_u(v) = g(u, v) \quad \text{for all} \quad v \in S_u.$$

In keeping with the terminology which we introduced earlier in the thesis, the univariate functional g_u is called the u-section of g.

A relation S is called reflexive if $(u, u) \in S$ for all $u \in \operatorname{dom} S$. The following original definition describes what it means for a relation S to be suitable for our purposes:

Definition 11.1 *Let V be a vector space over a field F, and let $S \subset V \times V$ be an arbitrary relation on V. We say that S* ***is compatible with*** *V if the image S_u is a subspace of V for all $u \in \operatorname{dom} S$. We say that S* ***is a suitable relation on*** *V if both of the following two axioms are satisfied:*

1. *S is reflexive.*

2. *S is compatible with V.*

A suitable relation $S \subset V \times V$ serves as the domain of a generalized inner product $g : S \to F$. Since S is reflexive, the expression $g(u,u)$ is well-defined for all $u \in \operatorname{dom} S$, and we can therefore consider whether the scalar $g(u,u)$ is invertible. Since S is compatible with V, the image S_u is a subspace of V for each $u \in \operatorname{dom} S$, and we can therefore consider whether the univariate functional $g_u : S_u \to F$ is linear. If the u-section g_u is a linear functional, we can express this concisely in dual-space notation by writing $g_u \in S_u^*$.

The following original definition specifies the axioms that must be satisfied in order for a bivariate functional $g : S \to F$ to be a generalized inner product on the vector space V:

Definition 11.2 *Let V be a vector space over a field F, let $S \subset V \times V$ be a suitable relation on V, and let $g : S \to F$ be an arbitrary bivariate functional. The functional g is called* ***definite*** *if $g(u,u) \neq 0$ for all $u \in \operatorname{dom} S \backslash \{0\}$. The functional g is called* ***unilinear*** *if $g_u \in S_u^*$ for all $u \in \operatorname{dom} S$. We say that g* ***is a generalized inner product on*** *V if both of the following two axioms are satisfied:*

1. *g is definite.*
2. *g is unilinear.*

Finally, if g is a generalized inner product on V and $(u,v) \in V \times V$ is arbitrary, we can assert that $(u,v) \in S$ by saying that the expression $g(u,v)$ ***is defined****.*

In summary, a generalized inner product on a vector space V is a definite, unilinear functional whose domain is a reflexive, V-compatible subset of $V \times V$. We can now define what it means for the vector space V to be a generalized inner product space:

Definition 11.3 *Let V be a vector space over a field F, let $S \subset V \times V$ be a suitable relation on V, and let $g : S \to F$ be a generalized inner product on V. We call the ordered pair (V,g) a* ***generalized inner product space****.*

Given a generalized inner product space (V,g), we can recover the suitable relation $S \subset V \times V$ directly from the functional g via $S = \operatorname{dom} g$. That is why we denote the generalized inner product space by the ordered pair (V,g) instead of the ordered triple (V,S,g). Similarly, we can recover the domain of S via

$$\operatorname{dom}^2 g = \operatorname{dom}(\operatorname{dom} g) = \operatorname{dom} S.$$

The following example shows that classical inner product spaces over the real numbers are indeed generalized inner product spaces in the sense defined above:

Example 11.4 (Real Inner Product Spaces) *Let $(V,\ \langle\bullet,\bullet\rangle)$ be a classical inner product space over the field $\mathbb{R}$. Define the relation $S := V \times V$ and note that $\operatorname{dom} S = V$. The relation S is clearly reflexive. Since $S_u = V$ for all $u \in \operatorname{dom} S$, the relation S is compatible with V. We conclude that S is a suitable relation on V. Define the bivariate functional $g : S \to F$ by $g(u,v) = \langle u, v\rangle$ for all $(u,v) \in S$. Since g is positive-definite, it follows that $g(u,u) > 0$ for all $u \in \operatorname{dom} S\backslash\{0\}$, which implies that g is definite. Since g is bilinear, the u-section g_u is linear for all $u \in \operatorname{dom} S$, which implies that g is unilinear. We conclude that g is a generalized inner product on V. In summary, the classical inner product space $(V,\ \langle\bullet,\bullet\rangle)$ is also a generalized inner product space in the sense defined above.*

Some generalized inner products possess a convenient normalization property which has no analogue over classical inner product spaces:

Definition 11.5 *Let (V, g) be a generalized inner product space over a field F. If*

$$g(u,u) = 1 \quad \textit{for all} \quad u \in (\operatorname{dom}^2 g)\backslash\{0\},$$

*we say that the generalized inner product g **is normalized**, and we also say that the generalized inner product space (V, g) **is normalized**.*

If $(V,\ \langle\bullet,\bullet\rangle)$ is a classical inner product space over $\mathbb{R}$, the functional $v \mapsto \langle v, v\rangle$ is a positive-definite quadratic form on V. By definition, a quadratic form is homogeneous of degree two and is therefore *nonconstant* on $V\backslash\{0\}$. This means that a *classical* inner product space *cannot* possess the normalization property defined above. By contraposition, any generalized inner product space which *is* normalized *cannot* be a classical inner product space!

The following example uses the asymptotic constructions of Chapter 6 to introduce a generalized inner product space which is *normalized* and therefore *nonclassical:*

Example 11.6 (Asymptotic Inner Product Spaces) *Let $X \subset \overline{\mathbb{R}}$ be an arbitrary subset, and let x_0 be a limit point of X. Define the function space $V := \mathbb{R}^X$, and define the proper subset $U := \mathrm{SNV}_{x_0}(X)$.*

Let S denote the set of all ordered pairs $(u,v) \in U \times V$ such that the asymptotic order relation $v(x) = \tilde{o}_L(u(x))$ as $x \to x_0$ holds for some constant $L \in \mathbb{R}$. By construction, $S \subset V \times V$ is a relation on V. Proposition 6.12 assures us that the constant L is uniquely *determined for each* $(u,v) \in S$. *Proposition 6.12 also implies that* $(u,v) \in S$ *if and only if*

$$\text{a-supp}\, v \subset_{x_0} \text{a-supp}\, u \quad \textit{and} \quad \lim_{\substack{x \rightsquigarrow x_0 \\ u}} \frac{v(x)}{u(x)} = L.$$

If $u \in U$ *and* $v = u$, *these two conditions are automatically satisfied with the constant* $L = 1$. *From this, it follows that* $(u,u) \in S$ *for all* $u \in U$. *This implies that* $\operatorname{dom} S = U$, *and shows that the relation* S *is reflexive. The inclusion of algebraic supports*

$$\text{a-supp}(\alpha_1 v_1 + \alpha_2 v_2) \subset \text{a-supp}\, v_1 \cup \text{a-supp}\, v_2 \quad \textit{for all} \quad \alpha_i \in F,\ v_i \in V$$

and the linearity of the supporting sequential limit as $x \underset{u}{\rightsquigarrow} x_0$ *together imply that the image* S_u *is a subspace of* V *for all* $u \in U$. *This shows that* S *is compatible with* V. *We conclude that* S *is a suitable relation on* V.

Define the bivariate functional $g : S \to \mathbb{R}$ *for all* $(u,v) \in S$ *by* $g(u,v) := L$, *where* $L \in \mathbb{R}$ *is the unique constant such that* $v(x) = \tilde{o}_L(u(x))$ *as* $x \to x_0$. *Proposition 6.12 implies that*

$$g(u,v) = \lim_{\substack{x \rightsquigarrow x_0 \\ u}} \frac{v(x)}{u(x)} \qquad \textit{for all} \quad (u,v) \in S.$$

In the special case $u \in \mathrm{LNV}_{x_0}(X)$, *the previous formula reduces to*

$$g(u,v) = \lim_{x \to x_0} \frac{v(x)}{u(x)}.$$

Since $g(u,u) = 1$ *for all* $u \in U$, *the bivariate functional* g *is normalized and therefore definite. In addition, the linearity of the supporting sequential limit as* $x \underset{u}{\rightsquigarrow} x_0$ *implies that the univariate functional* g_u *is linear for all* $u \in U$. *This shows that the bivariate functional* g *is unilinear. We conclude that* g *is a generalized inner product on* V. *In summary,* (V,g) *is a normalized generalized inner product space.*

The following definition introduces some original terminology and notation based on the previous example:

Definition 11.7 *Let* $X \subset \overline{\mathbb{R}}$ *be an arbitrary subset, and let* x_0 *be a limit point of* X. *Let* g *denote the generalized inner product which we defined on the function space* $\mathbb{R}^X$ *in Example 11.6. From now on, we will use the preferred notation*

$$[u,v]_{x_0} := g(u,v) \quad \textit{for all} \quad (u,v) \in \operatorname{dom} g.$$

We call $[\bullet, \bullet]_{x_0}$ *the* ***asymptotic inner product on*** $\mathbb{R}^X$ ***at*** x_0*, and we call* $(\mathbb{R}^X,\ [\bullet, \bullet]_{x_0})$ *the* ***asymptotic inner product space at*** x_0*.*

Given arbitrary functions $u, v \in \mathbb{R}^X$ such that $v(x) = \tilde{o}_L(u(x))$ as $x \to x_0$, Corollary 6.13 implies that the condition $u \in \mathrm{SNV}_{x_0}(X)$ is the *weakest* possible hypothesis which ensures the *uniqueness* of the constant L. If we were to *extend* the domain of the asymptotic inner product $[\bullet, \bullet]_{x_0}$ to a larger suitable relation on the function space $\mathbb{R}^X$, we would no longer be assured of obtaining a *unique* constant L from the asymptotic order relation $\tilde{o}_L$. This means that we have defined the asymptotic inner product space $(\mathbb{R}^X,\ [\bullet, \bullet]_{x_0})$ *at the maximal level of generality possible!* That is the reason we worked so hard to develop the basic theory of asymptotic analysis in one real variable at a higher level of generality than is customary.

11.2 Transitive and Multiplicative Properties

In this section, we will develop transitive and multiplicative properties for generalized inner product spaces. Although every classical inner product space is transitive, we will find that only nonclassical generalized inner product spaces can be multiplicative. We will also consider how the previously defined normalization property is related to these new transitive and multiplicative properties. We will conclude this section by showing that every asymptotic inner product space has the multiplicative property.

An arbitrary relation $S \subset V \times V$ is called transitive if $(u, v) \in S$ and $(v, w) \in S$ imply that $(u, w) \in S$. We can use this property of relations to define the following notion of transitivity for generalized inner product spaces:

Definition 11.8 *Let* (V, g) *be a generalized inner product space over a field* F*. If the suitable relation* $S := \operatorname{dom} g$ *is transitive, we say that the generalized inner product* g ***is transitive****, and we say that the generalized inner product space* (V, g) ***is transitive****.*

It is easy to construct examples of generalized inner product spaces which are transitive:

Example 11.9 *Every classical inner product space* $(V,\ \langle \bullet, \bullet \rangle)$ *over* $\mathbb{R}$ *is transitive since the suitable relation* $\operatorname{dom} \langle \bullet, \bullet \rangle = V \times V$ *is transitive.*

The following example of a transitive generalized inner product space is very useful in applications involving the *interpolation of linear functional data on abstract vector spaces:*

Example 11.10 (Linear Functionals) *Let V be a vector space over a field F. Let $\phi \in V^*\backslash\{0\}$ be a nontrivial linear functional, and define the nonempty proper subset $V_\phi := V\backslash \ker\phi$. The Cartesian product $V_\phi \times V$ is a suitable relation on V, and the functional $[\bullet,\bullet]_\phi : V_\phi \times V \to F$ defined by*

$$[u,v]_\phi := \frac{\phi(v)}{\phi(u)} \quad \text{for all} \quad (u,v) \in V_\phi \times V$$

is a generalized inner product on V. Since $[u,u]_\phi = 1$ for all $u \in V_\phi$, the generalized inner product $[\bullet,\bullet]_\phi$ is normalized. Since the relation $V_\phi \times V$ is transitive, the generalized inner product $[\bullet,\bullet]_\phi$ is also transitive. In summary, $(V,\ [\bullet,\bullet]_\phi)$ is a transitive, normalized, generalized inner product space.

The following definition introduces some original terminology based on the previous example:

Definition 11.11 *Let V be a vector space over a field F, and let $\phi \in V^*\backslash\{0\}$ be a nontrivial linear functional. Let $[\bullet,\bullet]_\phi$ denote the generalized inner product which we defined on V in Example 11.10. We call $[\bullet,\bullet]_\phi$ the **generalized inner product induced on V by ϕ**, and we call $(V,\ [\bullet,\bullet]_\phi)$ the **generalized inner product space induced by ϕ**.*

Some transitive generalized inner product spaces also possess a multiplicative property which has no analogue over classical inner product spaces:

Definition 11.12 *Let (V,g) be a transitive generalized inner product space over a field F. If*

$$g(u,w) = g(u,v) \cdot g(v,w) \quad \text{for all} \quad (u,v),\ (v,w) \in \operatorname{dom} g,$$

*we say that the generalized inner product g **is multiplicative**, and we say that the generalized inner product space (V,g) **is multiplicative**.*

In summary, if the generalized inner product space (V,g) is *transitive*, then $g(u,w)$ is defined whenever both $g(u,v)$ and $g(v,w)$ are defined. If the generalized inner product space (V,g) is *multiplicative*, then the three scalars $g(u,w)$ and $g(u,v)$ and $g(v,w)$ are related to each other via $g(u,w) = g(u,v) \cdot g(v,w)$.

The following proposition establishes that every multiplicative generalized inner product space is also normalized:

Proposition 11.13 *Let (V, g) be a generalized inner product space over a field F. If the generalized inner product g is multiplicative, then g is also normalized.*

Proof. Let $u \in (\operatorname{dom}^2 g)\backslash\{0\}$, and note that $(u, u) \in \operatorname{dom} g$ since the relation $\operatorname{dom} g$ is reflexive. Since g is multiplicative, it follows that $g(u, u) = g(u, u) \cdot g(u, u) = g^2(u, u)$. This means that $g(u, u)$ is idempotent, which implies that either $g(u, u) = 0$ or $g(u, u) = 1$. Since g is definite, we must have $g(u, u) = 1$. This shows that g is normalized. ∎

In summary, the *multiplicative* property in generalized inner product spaces presupposes the *transitive* property and implies the *normalization* property. The following example establishes that asymptotic inner product spaces are multiplicative:

Example 11.14 *Let $X \subset \overline{\mathbb{R}}$ be an arbitrary subset, and let x_0 be a limit point of X. Let $(\mathbb{R}^X,\ [\bullet, \bullet]_{x_0})$ be the asymptotic inner product space at x_0, and let $S := \operatorname{dom}[\bullet, \bullet]_{x_0}$ denote the corresponding suitable relation on $\mathbb{R}^X$. Recall that $\operatorname{dom} S = \mathrm{SNV}_{x_0}(X)$. If $(u, v) \in S$ and $(v, w) \in S$, then by definition, the asymptotic order relations $v(x) = \tilde{o}_L(u(x))$ as $x \to x_0$ and $w(x) = \tilde{o}_M(v(x))$ as $x \to x_0$ hold with constants $L := [u, v]_{x_0}$ and $M := [v, w]_{x_0}$. By arguing directly from the original definitions of the asymptotic order relations $\tilde{o}_L$ and $\tilde{o}_M$ using inequalities and epsilon neighborhoods of x_0, we can show that the asymptotic order relation $w(x) = \tilde{o}_{LM}(u(x))$ as $x \to x_0$ also holds.[1] Since $u \in \mathrm{SNV}_{x_0}(X)$ and $w \in \mathbb{R}^X$ by assumption, it follows that $(u, w) \in S$ by definition. This shows that the relation S is transitive. In addition,*

$$[u, w]_{x_0} := LM = [u, v]_{x_0} \cdot [v, w]_{x_0}$$

by definition. We conclude that the asymptotic inner product $[\bullet, \bullet]_{x_0}$ is multiplicative. In summary, the asymptotic inner product space $(\mathbb{R}^X,\ [\bullet, \bullet]_{x_0})$ is multiplicative, transitive, and normalized.

The multiplicative property of asymptotic inner product spaces will prove to be instrumental in showing that asymptotic sequences are in fact orthogonal sequences in a generalized sense. Towards that end, we will now study what it means for vectors to be orthogonal in a generalized inner product space.

[1] This is an excellent exercise for students of analysis. The main idea is to derive the inequality we want from the inequalities we are given by repeatedly using the classic trick of subtracting and adding an appropriate term.

11.3 Orthogonality in Generalized Inner Product Spaces

We will begin this section by defining an orthogonality relation on generalized inner product spaces. We will then use this relation to construct orthogonal complements. We will also show that this orthogonality relation is automatically transitive whenever the generalized inner product space is multiplicative. We will conclude this section by showing that the orthogonality relation in asymptotic inner product spaces is actually characterized by Landau's asymptotic order relation o.

The following definition describes the orthogonality relation and explains how to construct orthogonal complements of both single vectors and sets of vectors:

Definition 11.15 *Let (V, g) be a generalized inner product space over a field F. If $(u, v) \in \operatorname{dom} g$ and $g(u, v) = 0$, we say that u **is orthogonal to** v, and we write $u \perp v$. Given a single vector $u \in \operatorname{dom}^2 g$, we define the **orthogonal complement of** u, denoted by $u^\perp$, via*

$$u^\perp := \{v \in V : u \perp v\}.$$

*Given a set of vectors $U \subset \operatorname{dom}^2 g$, we define the **orthogonal complement of** U, denoted by $U^\perp$, via*

$$U^\perp := \bigcap_{u \in U} u^\perp.$$

Like the suitable relation $S := \operatorname{dom} g$, the orthogonality relation $\perp$ is a relation on the vector space V. Furthermore, the orthogonality relation $\perp$ is a subrelation of the relation S by definition.

Given $u \in \operatorname{dom} S$, we can reformulate the definition of the orthogonal complement $u^\perp$ as follows:

$$u^\perp := \{v \in V : u \perp v\} = \{v \in S_u : g_u(v) = 0\} = \ker g_u.$$

Since $u^\perp$ is the kernel of the univariate linear functional g_u, it follows that $u^\perp$ is a subspace of V. We thus have the following inclusion of subspaces:

$$u^\perp \subset S_u \subset V \quad \text{for all} \quad u \in \operatorname{dom} S.$$

Given $U \subset \operatorname{dom} S$, the orthogonal complement $U^\perp$ is by definition the intersection of subspaces of V and is therefore also a subspace of V. This yields the subspace inclusion $U^\perp \subset V$.

The following original proposition establishes that the *orthogonality relation* is transitive whenever the *generalized inner product* is multiplicative:

Proposition 11.16 *Let (V, g) be a generalized inner product space over a field F. If the generalized inner product g is multiplicative, then the orthogonality relation $\perp$ is transitive.*

Proof. Let $u, v, w \in V$ be arbitrary, and assume that $u \perp v$ and $v \perp w$. By definition, $g(u, v)$ and $g(v, w)$ are both defined, and $g(u, v) = g(v, w) = 0$. Since g is transitive, $g(u, w)$ is also defined. Since g is multiplicative, $g(u, w) = g(u, v) \cdot g(v, w) = 0^2 = 0$. It follows that $u \perp w$ by definition, which shows that $\perp$ is transitive. ■

Since *asymptotic inner products* are multiplicative, the previous proposition implies that the *orthogonality relation* is transitive on asymptotic inner product spaces. Recall that Landau's asymptotic order relation o is transitive, and arises as a special case of the author's asymptotic order relation $\tilde{o}_L$, namely when $L = 0$. The following example establishes that the transitive asymptotic order relation o in fact *characterizes* the transitive orthogonality relation in an asymptotic inner product space:

Example 11.17 *Let $X \subset \overline{\mathbb{R}}$ be an arbitrary subset, and let x_0 be a limit point of X. Let $(\mathbb{R}^X,\ [\bullet, \bullet]_{x_0})$ be the asymptotic inner product space at x_0,and let $S := \operatorname{dom}[\bullet, \bullet]_{x_0}$ denote the corresponding suitable relation on $\mathbb{R}^X$. Recall that $\operatorname{dom} S = \mathrm{SNV}_{x_0}(X)$, and let $u, v \in \mathbb{R}^X$ be arbitrary. By definition, the relation $u \perp v$ holds if and only if $(u, v) \in S$ and $[u, v]_{x_0} = 0$. This occurs if and only if $u \in \mathrm{SNV}_{x_0}(X)$ and the asymptotic order relation $v(x) = \tilde{o}_L(u(x))$ as $x \to x_0$ holds with $L = 0$. This is equivalent to the condition that $u \in \mathrm{SNV}_{x_0}(X)$ and $v(x) = o(u(x))$ as $x \to x_0$. We conclude that a function $u \in \mathrm{SNV}_{x_0}(X)$ is orthogonal to an arbitrary function $v \in \mathbb{R}^X$ in the asymptotic inner product space $(\mathbb{R}^X,$ $[\bullet, \bullet]_{x_0})$ if and only if $v(x) = o(u(x))$ as $x \to x_0$.*

The transitive property of the asymptotic orthogonality relation is the key to showing that asymptotic sequences are orthogonal sequences in the generalized sense. We will establish this important result in the next section.

11.4 Generalized Orthogonal Sequences and Series

In this section, we will use the notion of orthogonality developed in the previous section to define orthogonal sequences and orthogonal series over generalized inner product spaces. We will then illustrate the differences between orthogonal sequences in the classical sense and orthogonal sequences in the generalized sense by examining the structure of the resulting Gram matrix. Next, we will describe an algorithm which uses the generalized inner product to compute the Fourier coefficients of a vector with respect to an orthonormal sequence. After this, we will show that asymptotic sequences and series are precisely the orthonormal sequences and series in asymptotic inner product spaces. We will conclude this section by discussing orthogonal series over the tensor product of two generalized inner product spaces, thereby introducing the theoretical framework needed to justify the title of the thesis.

The following original definition explains what it means for a *sequence* of suitable vectors to be orthogonal (orthonormal) in a generalized inner product space, and also defines an orthogonal (orthonormal) *series* in this abstract context:

Definition 11.18 *Let (V, g) be a generalized inner product space over a field F. Let n be either a positive integer or infinity, and assume that $\{u_i\}_{i=0}^{n-1} \subset (\operatorname{dom}^2 g)\backslash\{0\}$. We say that the sequence $\{u_i\}_{i=0}^{n-1}$* ***is orthogonal*** *if*

$$u_i \perp u_j \quad \textit{for} \quad 0 \le i < j < n.$$

We say that the sequence $\{u_i\}_{i=0}^{n-1}$ ***is orthonormal*** *if $\{u_i\}_{i=0}^{n-1}$ is orthogonal and satisfies*

$$g(u_i, u_i) = 1 \quad \textit{for} \quad 0 \le i < n.$$

Given coefficients $\{c_i\}_{i=0}^{n-1} \subset F$, we say that the formal series

$$s_n := \sum_{i=0}^{n-1} c_i \cdot u_i$$

is orthogonal *if the sequence $\{u_i\}_{i=0}^{n-1}$ is orthogonal, and we say that the formal series s_n* ***is orthonormal*** *if the sequence $\{u_i\}_{i=0}^{n-1}$ is orthonormal. If s_n is an orthonormal series, we call the coefficients $\{c_i\}_{i=0}^{n-1}$ the* ***Fourier coefficients of*** *s_n* ***with respect to*** *$\{u_i\}_{i=0}^{n-1}$.*

In the degenerate case $n = 1$, it is *vacuously* true that the sequence $\{u_i\}_{i=0}^{n-1}$ is orthogonal. Note that if the generalized inner product space is *normalized*, then every *orthogonal* sequence is automatically *orthonormal*.

In order to illustrate the essential differences between orthogonal sequences in *classical* inner product spaces over $\mathbb{R}$ and orthogonal sequences in *generalized* inner product spaces over F, let us consider the $n \times n$ Gram matrix

$$G := [g(u_i, u_j)]_{i,j=0}^{n-1}.$$

If the sequence $\{u_i\}_{i=0}^{n-1}$ is orthogonal in the *classical* sense, the matrix G is *diagonal*; however, if the sequence $\{u_i\}_{i=0}^{n-1}$ is orthogonal in the *generalized* sense, the matrix G is merely *lower-triangular*—it consists entirely of zeros *above* the main diagonal, and the entries *below* the main diagonal do not even need to be defined. Since every diagonal matrix is clearly lower-triangular, every orthogonal sequence in the classical sense is also an orthogonal sequence in the generalized sense. What about the converse? The Gram matrix G is always *symmetric* in the classical case due to the symmetry of the classical inner product. Since every symmetric lower-triangular matrix is diagonal, every orthogonal sequence in the generalized sense is also an orthogonal sequence in the classical sense when we are working over a classical inner product space. We thus arrive at the following two conclusions:

1. The generalized notion of orthogonal sequences can be *strictly weaker* than the classical notion when we are working over generalized inner product spaces.

2. The classical and generalized notions of orthogonal sequences always *coincide* when we are working over classical inner product spaces.

The following algorithm demonstrates how to exploit the triangular structure of the Gram matrix to calculate the Fourier coefficients of an orthonormal series in a generalized inner product space:

Algorithm 11.19 (Fourier Coefficient) *Let (V, g) be a generalized inner product space over a field F. Let n be a positive integer, let $\{c_i\}_{i=0}^{n-1} \subset F$ and $\{u_i\}_{i=0}^{n-1} \subset (\mathrm{dom}^2\, g)\backslash\{0\}$, and assume that the series*

$$v := \sum_{i=0}^{n-1} c_i \cdot u_i$$

is orthonormal. For $0 \leq i \leq n$, let s_i denote the i-th partial sum of the series, and define the corresponding remainder by $r_i := v - s_i$. By convention, we define $s_0 := 0$ and $r_0 := v$. We can calculate the Fourier coefficients of v with respect to $\{u_i\}_{i=0}^{n-1}$ by iterating the following equation for $i = 0, 1, \ldots, n-1$:

$$c_i := g(u_i, r_i).$$

We can easily modify the above algorithm to handle *orthogonal* series as well as *orthonormal* series. In the more general case, we calculate the coefficients by iterating the following equation for $i = 0, 1, \ldots, n-1$:

$$c_i := \frac{g(u_i, r_i)}{g(u_i, u_i)}.$$

This extended algorithm illustrates the importance of both the definiteness and the unilinearity of g, and has the following useful consequence:

Proposition 11.20 *Let (V, g) be a generalized inner product space over a field F. Let n be a positive integer, and assume that $\{u_i\}_{i=0}^{n-1} \subset (\operatorname{dom}^2 g)\backslash\{0\}$. If $\{u_i\}_{i=0}^{n-1}$ is an orthogonal sequence in (V, g), then $\{u_i\}_{i=0}^{n-1}$ is a linearly independent set in V.*

Given a suitable linearly independent set $\{v_i\}_{i=0}^{n-1} \subset V$, it is possible to adapt the classical Gram-Schmidt orthogonalization algorithm in order to construct an orthogonal sequence $\{u_i\}_{i=0}^{n-1}$ in the generalized inner product space (V, g). The idea is to refine the input sequence $\{v_i\}_{i=0}^{n-1}$ in an iterative fashion using a sequence of orthogonal projections to produce an output sequence $\{u_i\}_{i=0}^{n-1}$ which is orthogonal in (V, g) and satisfies

$$\operatorname{span}_F\{u_i\}_{i=0}^{m} = \operatorname{span}_F\{v_i\}_{i=0}^{m} \quad \text{for} \quad 0 \le m < n.$$

The Gram-Schmidt orthogonalization algorithm for generalized inner product spaces can be used *to generate asymptotic sequences* in asymptotic inner product spaces. It can also be used *to generate various kinds of interpolation bases* in function spaces—such as Taylor interpolation bases, Newton interpolation bases, and Hermite interpolation bases, for example. Unfortunately, the details of the generalized Gram-Schmidt algorithm—although completely elementary—lie outside of the scope of the thesis.

Note that Fourier Coefficient Algorithm 11.19 is based on *exactly* the same iteration that we used to calculate the coefficients of an asymptotic expansion in Proposition 6.18! The following example establishes that asymptotic sequences and series are in fact orthonormal sequences and series in asymptotic inner product spaces—and conversely:

Example 11.21 *Let $X \subset \overline{\mathbb{R}}$ be an arbitrary subset, let x_0 be a limit point of X, and let $(\mathbb{R}^X,\ [\bullet, \bullet]_{x_0})$ be the asymptotic inner product space at x_0. Let n be either a positive integer or infinity, and assume that $\{u_i\}_{i=0}^{n-1} \subset \mathrm{SNV}_{x_0}(X)$. By definition, $\{u_i(x)\}_{i=0}^{n-1}$ is an* *asymptotic sequence as $x \to x_0$ if and only if*

$$u_{i+1}(x) = o(u_i(x)) \quad as \quad x \to x_0 \quad for \quad 0 \le i < n-1.$$

Since the asymptotic order relation o characterizes the orthogonality relation $\perp$, the previous condition is equivalent to

$$u_i \perp u_{i+1} \quad for \quad 0 \le i < n-1.$$

Since the asymptotic orthogonality relation is transitive, the previous condition implies that

$$u_i \perp u_j \quad for \quad 0 \le i < j < n.$$

This shows that $\{u_i\}_{i=0}^{n-1}$ is an orthogonal sequence. Since the asymptotic inner product is normalized, the sequence $\{u_i\}_{i=0}^{n-1}$ is orthonormal as well. Given coefficients $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}$, the asymptotic series

$$s_n := \sum_{i=0}^{n-1} c_i \cdot u_i$$

is also orthonormal by definition. The converse is clearly true as well: Every orthonormal sequence and orthonormal series over the asymptotic inner product space $(\mathbb{R}^X,\ [\bullet, \bullet]_{x_0})$ is also an asymptotic sequence and asymptotic series as $x \to x_0$.

Now that we have considered orthogonal sequences and series over *one* generalized inner product space, we are ready to extend the notion of orthogonality to series constructed over the tensor product of *two* generalized inner product spaces. This means that at long last, we can justify the title of the thesis! We will now explain the sense in which both dual asymptotic expansions and Geddes series expansions constitute generalized orthogonal series.

Given two *classical* inner product spaces $(U,\ \langle \bullet, \bullet \rangle_U)$ and $(V,\ \langle \bullet, \bullet \rangle_V)$ over $\mathbb{R}$, there is a standard technique for constructing a classical inner product on the tensor product space $U \otimes V$. It is in fact possible to carry out a similar construction on the tensor product of two *generalized* inner product spaces (U, g) and (V, h) over a field F; however, for the sake of brevity, we will omit the details of these constructions and simply state the following consequence, which gives us a way to construct orthogonal (orthonormal) series over the tensor product space $(U \otimes V,\ g \otimes h)$.

Proposition 11.22 *Let (U, g) and (V, h) be generalized inner product spaces over a field*

F. Let n be either a positive integer or infinity. Assume that $\{c_i\}_{i=0}^{n-1} \subset F\backslash\{0\}$, that $\{u_i\}_{i=0}^{n-1} \subset (\operatorname{dom}^2 g)\backslash\{0\}$, and that $\{v_i\}_{i=0}^{n-1} \subset (\operatorname{dom}^2 h)\backslash\{0\}$. If the sequence $\{u_i\}_{i=0}^{n-1}$ is orthogonal in (U, g) and the sequence $\{v_i\}_{i=0}^{n-1}$ is orthogonal in (V, h), then the formal tensor product series

$$s_n := \sum_{i=0}^{n-1} c_i \cdot (u_i \otimes v_i)$$

is orthogonal in $(U \otimes V,\ g \otimes h)$. Similarly, if the sequences $\{u_i\}_{i=0}^{n-1}$ and $\{v_i\}_{i=0}^{n-1}$ are both orthonormal, then the formal series s_n is orthonormal.

The following example applies this extended notion of orthonormal series to dual asymptotic series:

Example 11.23 *Let $X, Y \subset \overline{\mathbb{R}}$ be arbitrary subsets, and let x_0 and y_0 be limit points of X and Y, respectively. Let $(\mathbb{R}^X,\ [\bullet, \bullet]_{x_0})$ denote the asymptotic inner product space at x_0 and let $(\mathbb{R}^Y,\ [\bullet, \bullet]^{y_0})$ denote the asymptotic inner product space at y_0. Let n be either a positive integer or infinity, and assume that $\{c_i\}_{i=0}^{n-1} \subset \mathbb{R}\backslash\{0\}$, that $\{u_i\}_{i=0}^{n-1} \subset \mathrm{SNV}_{x_0}(X)$, and that $\{v_i\}_{i=0}^{n-1} \subset \mathrm{SNV}_{y_0}(Y)$. If the formal tensor product series*

$$s_n(x, y) := \sum_{i=0}^{n-1} c_i \cdot u_i(x)\, v_i(y)$$

is a dual asymptotic series as $x \to x_0$ or $y \to y_0$, then by definition, $\{u_i(x)\}_{i=0}^{n-1}$ is an asymptotic sequence as $x \to x_0$ and $\{v_i(y)\}_{i=0}^{n-1}$ is an asymptotic sequence as $y \to y_0$. This means that the sequence $\{u_i\}_{i=0}^{n-1}$ is orthonormal in $(\mathbb{R}^X,\ [\bullet, \bullet]_{x_0})$ and the sequence $\{v_i\}_{i=0}^{n-1}$ is orthonormal in $(\mathbb{R}^Y,\ [\bullet, \bullet]^{y_0})$. We conclude that the formal series $s_n(x, y)$ is an orthonormal series in the generalized sense described above.

As a consequence of the previous example, all dual asymptotic expansions—including, for example, l'Hôpital dual asymptotic expansions—are generalized orthonormal series. This means that all of the tensor product identities which arise directly from dual asymptotic expansions are in fact generalized orthonormal series expansions in two variables. It also means that Neumann's addition formula for the Bessel function of the first kind of order zero is a generalized orthonormal series expansions in two variables.

In a similar fashion, we can show that every formal Geddes series

$$s_n := \sum_{i=0}^{n-1} \Omega_{(\phi_i, \psi_i)} r_i = \sum_{i=0}^{n-1} \frac{(\Psi_i r_i) \otimes (\Phi_i r_i)}{\Phi_i \Psi_i r_i}$$

which solves the strong $\Pi_n := (\boldsymbol{\phi}_n; \boldsymbol{\psi}_n)$ interpolation problem on a linear configuration (U, V, W) over a field F is also a generalized orthogonal series expansion. In this context, we must use the *n-fold generalized inner product spaces* $(U,\ \{g_i\}_{i=0}^{n-1})$ and $(V,\ \{h_i\}_{i=0}^{n-1})$, where $g_i := [\bullet, \bullet]_{\phi_i}$ and $h_i := [\bullet, \bullet]^{\psi_i}$ are the generalized inner products induced on U and V by the linear functionals ϕ_i and ψ_i, respectively. Unfortunately, the details of these constructions lie far outside the scope of the thesis.

11.5 Other Applications of Generalized Inner Products

Generalized inner product spaces in the sense of the thesis have many interesting applications which span a variety of different areas of mathematics. For example, we can characterize the radical elements of a finite field extension in terms of orthogonal sequences with respect to an appropriate generalized inner product. We can also express the nilradical of a finite-dimensional real commutative algebra as the orthogonal complement of the multiplicative group—with respect to an appropriate generalized inner product. We can even develop new kinds of non-Riemannian geometries in which hypercomplex-differentiable functions become conformal maps with respect to an appropriate generalized inner product.

Since p-adic expansions (and more generally, ideal-adic expansions) are essentially ring-theoretic analogues of asymptotic expansions, we can apply generalized inner products to these areas of mathematics as well. This requires that we develop the notion of a generalized inner product *module over a ring with respect to an ideal.*

These are merely the applications which the author has developed himself. Imagine all the wonderful possibilities that would spring into being if the collective strength of the mathematical community at large were brought to bear on this emerging area of research![2]

[2] The author is now accepting invitations to lecture on generalized inner product spaces at research seminars—not to mention birthday parties, debutante balls, high school proms, and weddings! Research seminars will receive priority service, of course.

Chapter 12

Reflections and Plans

In this final chapter, we will reflect upon the original contributions of the thesis with an emphasis on recurring themes. We will conclude the chapter—and the thesis—in the traditional way by outlining a number of promising possibilities for future research, thus ending our journey with the promise of a new adventure.

12.1 Reflections on Past Research Accomplishments

We already did quite a bit of reflecting on the main results of the thesis in the previous two chapters, which introduced and applied unifying abstractions pertaining to natural tensor product interpolation and generalized orthogonal series expansions. The author will now offer a few brief reflections on miscellaneous matters.

12.1.1 Asymptotic versus Abstract Splitting Operators

We have seen how important special cases of the *asymptotic* splitting operator can be realized by the *abstract* splitting operator for an appropriate choice of linear functionals. It is tempting to think that the abstract splitting operator, by virtue of its abstraction, supersedes its precursor, the asymptotic splitting operator. The author believes that this is a temptation which should be resisted rather than indulged since each of these two approaches has its own unique merits.

For example, the essence of the definition of the asymptotic splitting operator can be conveyed by a simple one-line equation expressed in terms of ordinary one-dimensional limits. This uses ideas which are fully accessible to students who have mastered one year of calculus. The abstract splitting operator, in contrast, has more formidable prerequisites based on abstract multilinear algebra and interpolation theory, and thus requires a more patient introduction. Although the asymptotic splitting operator is easier to define, we had to work much harder to develop a complete structure theory for the operator. In contrast, once we finally fulfilled all the prerequisites needed to define the abstract splitting operator, we found that its structure theory followed quickly and easily. In short, we are confronted with trade-offs, and there is a price to be paid either way: We can *pay now* (in the abstract case) or we can *pay later* (in the asymptotic case)!

It is also interesting to note that the asymptotic splitting operator has a more *dynamic* quality than the abstract splitting operator, which is more *static* in nature. What the asymptotic splitting operator actually does depends on the properties of the function to which it is applied—the number of times we invoke l'Hôpital's rule, for example, depends on the multiplicities of the zero lines through the splitting point. In contrast, the abstract splitting operator always does exactly the same thing to every function once the linear functionals which induce the splitting operator have been chosen.

There is another reason *not* to discard the asymptotic splitting operator in favor of the abstract splitting operator: Neither asymptotic analysis nor interpolation theory is entirely contained in the other. Although the thesis makes a convincing case that there is a substantial overlap between these two fields—for example, asymptotic expansions at a point in the domain of a function perform Taylor interpolation at that point—it is also the case that there are problems in asymptotic analysis which *cannot* be handled by the methods of interpolation theory, and conversely. For example, asymptotic analysis allows us to study the behavior of a function at a point where the function is not even defined—namely, at a limit point outside of the domain. The author's final recommendation is to enjoy and benefit from the overlap between these two fields as much as possible, but to remember that asymptotic analysis and interpolation theory are also separate and independent subjects.

12.1.2 An Ode to Labor-Saving Devices

The use of "labor-saving devices" has been a recurring theme throughout the thesis. Here is a list of the devices we have employed:

1. We developed the notion of the *generalized dual space*, which abstracts the essential properties of both algebraic and topological dual spaces, in order to get three mathematical results for the price of one in applications of dual spaces to tensor products.

2. We exploited the *universal property* of tensor products to easily define linear operators on tensor product spaces, thereby obviating the need to show that the value of the operator does not depend on the choice of the representation of the tensor product input.

3. We exploited *duality* in tensor product spaces to develop rigorous and yet concise treatments of similar results for each of the two factors of a tensor product space.

4. We developed *invariants for binormal forms* of tensor product expressions in order to exploit the labor already performed in reducing a tensor product expression to binormal form.

5. We developed *logical* tensor products of *logical* parametric extensions in order to use a new proof technique which exploits the recurring structure of the logical propositions which arise in the theory of l'Hôpital dual asymptotic expansions.

6. We exploited the *symmetry* of functions on squares in order to simplify the iterative algorithm for generating l'Hôpital dual asymptotic expansions.

7. We exploited *previously developed theory* to minimize the need for direct calculation when deriving and proving tensor product identities by hand.

8. Conversely, we exploited the availability of *computer-algebra systems* such as Maple to perform direct calculations, thereby minimizing the theoretical prerequisites when deriving and proving tensor product identities with machine assistance.

9. We used the *remainder technique of Donald Thomas* to exploit existing univariate remainder theories when developing new bivariate remainder theories for natural tensor product interpolation.

This shows that in mathematics as in life, opportunities to "work smarter, not harder" are indeed plentiful!

12.2 Plans for Future Research

This section briefly discusses a number of promising possibilities for future research based on the methods of the thesis. The author has already done some preliminary work in most of these areas.

12.2.1 Examples and Conjectures in Convergence Theory

The author has performed some preliminary studies on the uniform convergence of the Geddes series which solves the problem of interpolation on the lines of a two-dimensional grid. Through a variety of computational experiments—supplemented by a rigorous mathematical analysis of specific examples—the author has identified a wide variety of behaviors ranging from uniform convergence over a compact rectangle to pointwise almost-everywhere divergence. Guided by the these experimental results and mathematical analyses, the author has formulated the following plausible conjecture:

Conjecture 12.1 *The Geddes series described above converges uniformly to a given function over a compact rectangle whenever both of the following conditions hold:*

1. *The function is continuous on the compact rectangle.*

2. *The union of the grid lines is dense in the rectangle.*

The author has also performed some preliminary studies of the *rate* of uniform convergence. These studies consider a simple infinite class of examples with d but not $d+1$ continuous derivatives for $d = -1, 0, 1, 2 \ldots$. The author conjectures that the uniform error for this class of examples is $\Theta(1/n^{d+1})$ after n iterations, which means that the *smoother* functions in this class have series expansions with *faster* rates of uniform convergence.[1] This conjecture is supported by mathematical analysis for $d = -1$ (piecewise-continuity) and $d = 0$ (continuity), and by strong computational evidence for $d = 1, 2, 3$.

The author also proposes to study these convergence questions over the class of *positive definite functions.* Bochner's theorem characterizes these functions in terms of the Fourier transforms of Borel measures; this material can be found in [CL00]. The author has developed a potentially useful reformulation of Bochner's characterization in terms of Gram determinants; the many interesting properties of Gram determinants are discussed in [Dav63]. It may well be easier to develop a preliminary theory of convergence over the well-studied class of positive definite functions—as an interesting special case—before proceeding on to the general case of arbitrary continuous functions.

[1] The idea that a higher degree of smoothness leads to a faster rate of uniform convergence is a recurring theme in classical approximation theory. In one variable, for example, the Jackson theorems incorporate the order of differentiability into bounds on the error for the best uniform approximation by trigonometric or algebraic polynomials; see [Ach56], [Zyg59], [Riv69], and [Che82].

12.2.2 Symbolic-Numerical Algorithms for Tensor Product Approximation

The author proposes to use the Newton paradigm for natural tensor product interpolation—which is embodied by the Geddes series—to develop efficient algorithms for uniform approximation by natural tensor products.[2] Preliminary computational evidence strongly supports that these iterative algorithms are adaptive, numerically stable, and converge rapidly under reasonable smoothness assumptions.

The author first introduced a preliminary version of the proposed approximation algorithms in his 61 page doctoral thesis proposal: *An Adaptive Algorithm for Uniform Approximation by Tensor Products: With Applications to Multiple Integrals and Integral Equations* [Cha00]. This proposal presents some of the underlying interpolation theory, describes the implementation details of the author's first prototype of the approximation algorithm, and summarizes the results of early computational experiments.

It seems feasible to develop approximation algorithms based on the methods of the thesis at several levels of generality: first for *two real variables*, then for *two vector variables* in Euclidean space, and finally for *any number of real or vector variables.* The problem of ordinary interpolation on the lines of a two-dimensional grid is of particular interest since it makes minimal computational demands.

Recall that the examples of *exactly d-times continuously differentiable* functions discussed in the previous subsection have a conjectured uniform error of $\Theta(1/n^{d+1})$, which constitutes a *sublinear* rate of uniform convergence. The author's preliminary computational studies strongly suggest that the rate of uniform convergence for *real-analytic* functions is in fact *linear*—meaning that the uniform error eventually decreases by a constant factor with each successive iteration, resulting in rapid convergence of the Geddes series.

Since every function of two variables can be decomposed uniquely into the sum of a *symmetric* function and an *antisymmetric* function, we can always reduce the general bivariate approximation problem to symmetric and antisymmetric special cases. Consequently, we can develop specialized algorithms which approximate symmetric functions by symmetric natural tensor products and antisymmetric functions by antisymmetric natural tensor products. For the case of two real variables, the author has already implemented the symmetric approximation algorithm as a *hybrid symbolic-numerical method* in the Maple

[2] Performing function approximation by means of a convergent interpolatory process is another recurring theme in classical approximation theory; see [Dav63] and [SV90].

computer algebra system. Based on the strong computational evidence produced by this implementation, the author formulates the following conjecture:

Conjecture 12.2 *If we apply the symmetric approximation algorithm to a symmetric continuous function f on the symmetrically positioned unit square* $[0,1] \times [0,1]$, *the absolute error function* $|r_n|$ *attains its maximum value on the axis of symmetry, namely, the diagonal line* $y = x$, *for all sufficiently large* n. *This means that*

$$\max_{0 \leq x,y \leq 1} |r_n(x,y)| = \max_{0 \leq x \leq 1} |r_n(x,x)| .$$

This diagonal conjecture reveals an added benefit of symmetry that we can exploit to substantially reduce the computational cost of the symmetric algorithm: We can empirically estimate the uniform error over the *two-dimensional* unit square by sampling over the *one-dimensional* diagonal, *thereby cutting the dimension of the error-estimation problem in half.* For example, if we want to estimate the uniform error by sampling the *symmetric* absolute error function $|r_n|$ at the midpoints of the cells of an $n \times n$ grid, the diagonal conjecture reduces the number of function evaluations from $\frac{1}{2}n(n-1)$ (sampling both on and above the diagonal) to $n-1$ (sampling on the diagonal only); this reduction from $O(n^2)$ to $O(n)$ function evaluations is a substantial savings!

The author has studied several strategies for selecting the parameters of the underlying grid. In the author's experience, a good strategy produces an approximation which is numerically stable under floating-point evaluation, whereas a deliberately bad strategy can produce profound numerical instability when applied to particular functions. The author's preliminary Maple implementation uses a *maximum magnitude strategy* that not only generates a numerically stable approximation—it also *generates an adaptive grid* in the domain of the given function.[3] Thus, a happy by-product of the approximation algorithms proposed here is *an algorithm for the automatic generation of adaptive grids!*

The author proposes to extend these approximation algorithms to functions of *two vector variables* in Euclidean space. We can produce general natural tensor product approximations in *any number of real or vector variables* from these special natural tensor product approximations in *two vector variables* by an efficient recursion based on simple dimension-splitting; the depth of this recursion is logarithmic in the number of original vari-

[3] A grid in the domain of a given function is called adaptive if the grid lines occur closer together where the function has steeper slopes. For example, the lines of an adaptive grid would be spaced closer together in a region where the function oscillates rapidly and spaced farther apart in a region where the function is nearly constant.

ables. Note that this recursive approach to multivariate approximation by tensor products is inherently well-suited for implementation on parallel computer architectures.

12.2.3 New Adaptive Algorithms for Evaluating Multiple Integrals

The author proposes to use the tensor product approximation algorithms of the previous subsection to develop versatile new symbolic-numerical methods for evaluating both definite and indefinite integrals in any number of dimensions.[4] The key idea is to exploit the separation of variables in order to reduce the dimension of the integration problem. The author has already developed a promising Maple prototype of these quadrature methods in *two* dimensions, and believes these techniques can be successfully applied in *higher* dimensions as well. These quadrature methods inherit all the desirable properties of the approximation algorithms on which they are based: They are iterative, adaptive, numerically stable, and converge rapidly under reasonable smoothness assumptions. In addition, we readily obtain constructive, rigorous error estimates for the approximate values of the integrals by a simple and direct application of the appropriate natural tensor product remainder theory.

Using his prototype Maple implementation, the author has amassed a body of computational evidence which demonstrates that the method works well on a test suite consisting of *real-analytic functions*. In addition, the author has performed computational experiments—with favorable results—in order to extend this method to Lebesgue-integrable integrands $\phi(x, y)$ which are *real-analytic except at an isolated singularity*. The idea is to use asymptotic analysis to remove the singularity from the original integrand $\phi(x, y)$ and transfer the singularity to a product of integrable weight functions $w(x)\,w(y)$. Assuming our conjecture that the Geddes series converges uniformly for *continuous functions* on compact rectangles containing dense grids, this reformulation of the problem will allow us to approximate the new *continuous integrand*

$$f(x, y) := \frac{\phi(x, y)}{w(x)\,w(y)}$$

using the approximation algorithms already proposed. This in turn will allow us to ap-

[4]Integration via function approximation is a classical theme in numerical analysis. For example, Clenshaw-Curtis quadrature in one dimension is based on a Chebyshev series approximation of the integrand; see [Gen72a] and [Gen72b] for the implementation details of this scheme, and [Sny66] and [Riv90] for a discussion of the underlying theory.

proximate the value of the double integral

$$\int_0^1 \int_0^1 \phi(x, y)\, dx\, dy = \int_0^1 \int_0^1 f(x, y)\, w(x)\, w(y)\, dx\, dy$$

in a straightforward way.[5]

The author proposes to extended these approximate integration techniques via recursion in order to evaluate multiple integrals over hyperrectangles in any number of dimensions d. Exploiting the coarse-grain parallelism inherent to this approach may yield a new way to break the so-called "curse of dimensionality" that afflicts problems in multidimensional integration. The approach based on the thesis, like the approaches studied in [NR97] and [GG98], seeks to mitigate the effects of this curse in higher dimensions by invoking the "smoothness" of the integrand—which in this context means that the higher-order partial derivatives exist *and satisfy reasonable bounds*.

ACM TOMS Algorithm 698 is a more traditional adaptive multidimensional integration method which has been implemented for moderate dimensions ($2 \leq d \leq 15$); the algorithm is described, analyzed, and tested in [BEG91a], and a freely-available FORTRAN 77 implementation is documented in [BEG91b].[6] Algorithm 698 is also available in Maple (as of Version 8) via the command

```
evalf( Int( ..., method = _cuhre ) ).
```

Comparisons with existing algorithms and software such as this will be very helpful in accurately assessing the merits of the author's proposed multiple integration techniques.

12.2.4 Improved Algorithms for Solving Linear Integral Equations

We can apply the tensor product approximation algorithms proposed here to the problem of solving linear integral equations by indirect methods, thereby building on the previous work

[5] In general, approximate integration with high-accuracy in the presence of a singularity can be carried out much more efficiently if we first apply symbolic techniques in order to reformulate the problem in a neighborhood of the singularity. Maple has used this approach to handle singular integrands in one variable with great success for years; see [Ged86] and [GF92] for details.

[6] Software implementations of all the high-quality algorithms published by the Association for Computing Machinery (ACM) in the refereed *Transactions on Mathematical Software (TOMS)* can be found in the *Collected Algorithms (CALGO)* of the ACM, which are freely available online (http://www.acm.org/calgo/contents).

of Harry Bateman. Recall that in 1922, Bateman published a practical method for solving linear Fredholm integral equations of the second kind [Bat22]. Bateman's method uses the Lagrange paradigm for natural tensor product interpolation to uniformly approximate a *continuous* kernel by solving the interpolation problem on the lines of a two-dimensional grid. This results in a much simpler integral equation which we can solve exactly by the methods of linear algebra; furthermore, if we approximate the original kernel sufficiently well, the solution of the simplified problem will be a good approximation to the solution of the original problem.

The author proposes to make the following substantial improvements to Bateman's original method:

1. We can apply the Newton paradigm for natural tensor product interpolation instead of the Lagrange paradigm in order to create an iterative method which is much more efficient.

2. By applying our understanding of how to select the parameters of the underlying grid, we can produce a method which is also adaptive and numerically stable, hence more robust and reliable.

3. Since we know how to construct rigorous error estimates for a natural tensor product approximation of the kernel, we can develop a method which certifies the accuracy of the approximate solution to the original integral equation.

Can we apply similar methods to Volterra integral equations? Although it is true that every Volterra integral equation with a *continuous* kernel can be reformulated as a Fredholm integral equation with a *piecewise-continuous* kernel, this is of no help to us; a counterexample shows that the Geddes series *need not converge* in the case of a piecewise-continuous kernel. In order to apply the techniques of the thesis to Volterra integral equations, will have to be more creative!

12.3 Closing Remarks

The author wishes to thank the reader for making a substantial commitment of time and energy by studying this work, and sincerely hopes that the reader has shared the author's joy in making this epic mathematical journey! The author warmly welcomes thoughtful questions and constructive criticisms, as well as opportunities to collaborate with other mathematicians interested in further exploring some of the ideas proposed here.[7] Until our travels bring us together once again, journey well, dear reader—journey well.

[7] Please feel free to write to me at my permanent, lifetime e-mail address:

`fwchapman@alumni.uwaterloo.ca`

Appendix A

Using Maple to Derive and Prove Identities

We recall that a tensor product identity is an identity of the form

$$f(x, y) \equiv \sum_{i=0}^{n-1} g_i(x)\, h_i(y).$$

This appendix supplements the material in Chapter 9, *Algorithms for Deriving and Proving Identities*. The first section of this appendix includes the full code for a complete derivation and proof system which the author has implemented in Version 8 of Maple. This simple but effective system consists of two Maple routines called `derive` and `prove`; these two routines are based on the two-phase design of Weak Identity Algorithm 9.6.[1] The second section of this appendix includes examples which illustrate how to use Maple to automatically derive and prove various tensor product identities—including all of the identities that we previously derived and proved by hand in Chapter 9.

A.1 Maple Source Code

This section documents the intended usage of the `derive` and `prove` routines, and discusses salient implementation details. The Maple code for these routines is included in its entirety in the hope of encouraging experimentation by the reader.

[1] Please be aware that although the Maple implementation and the algorithm specification share the same essential design, they do use slightly different notation.

A.1.1 Maple Routine: `derive(expr, eq1, eq2, n)`

The argument `expr` is the expression $f(x, y)$ on the left-hand side of the tensor product identity. The argument `eq1` is an equation of the form $x = x_0$, which indicates that the variable x approaches the limit point x_0. Similarly, the argument `eq2` is an equation of the form $y = y_0$, which indicates that the variable y approaches the limit point y_0. The argument `n` is a positive integer n which specifies the desired number of terms in the tensor product on the right-hand side of the identity.

The `derive` routine returns the unique weak dual asymptotic expansion of f to n terms at (x_0, y_0), assuming that this expansion exists. The successful completion of this routine does *not* constitute a proof of existence—it merely generates the only possible *candidate*, assuming existence. When the weak dual asymptotic expansion does exist to n terms, it furnishes the rank-n tensor product on the right-hand side of the identity.

If Maple is unable to evaluate a particular one-dimensional limit—for example, because the limit does not exist over the real numbers, or because the limit exists only from one side—the `derive` routine returns `FAIL`. This indicates that the point (x_0, y_0) is unsuitable. In that case, we simply select a different point and try again. The examples given below illustrate a variety of techniques that easily overcome this difficulty.

Here is the complete code for the `derive` routine:

```
derive := proc(expr::algebraic, eq1::name=algebraic,
               eq2::name=algebraic, n::posint)
  local x, x0, y, y0, r, i, q, X, Y, t;
  x, x0 := lhs(eq1), rhs(eq1);  # parse eq1
  y, y0 := lhs(eq2), rhs(eq2);  # parse eq2
  r := unapply(expr, x, y);  # initial remainder
  for i from 0 to n-1 do
#   Generate the next term from the current remainder
#   via the asymptotic splitting operator at (x0,y0):
    q := r(x,Y)*r(X,y)/r(X,Y);  # intermediate quotient
    t := limit(limit(q, X=x0), Y=y0);  # next term
    if has(t,limit) then return FAIL end if;
    t := factor(t);  # express as rank-1 tensor product
    r := unapply(r(x,y) - t, x, y);  # next remainder
  end do;
  expr - r(x,y);  # return rank-n tensor product
```

```
end:
```

A call to the `derive` routine is typically followed by a call to the `prove` routine, which is discussed in the next subsection. Note that alternate proof criteria may apply in special circumstances; for example, Maple's `expand` command can be used to prove identities involving polynomials.

A.1.2 Maple Routine: `prove(expr, eq1, eq2, n)`

The argument `expr` is the remainder $r(x, y) := f(x, y) - s(x, y)$, where $f(x, y)$ denotes the closed-form expression on the left-hand side of the identity and $s(x, y)$ denotes the tensor product expression on the right-hand side of the identity. The arguments `eq1` and `eq2` are equations of the form $x = x_0$ and $y = y_0$, and the argument `n` is a positive integer n.

If r solves the hyperbolic eigenproblem of order n with homogeneous Cauchy data on $x = x_0$ and $y = y_0$, the `prove` routine returns the value `true` and calculates the eigenvalue λ as an additional confirmation. By Homogeneous Hyperbolic Eigenproblem Theorem 9.2, verifying these conditions proves that r is the zero function, which in turn proves that the tensor product identity $f = s$ holds.

In practice, the conditions which define the homogeneous hyperbolic eigenproblem are easy to verify. If Maple cannot verify one of the multiplicity conditions for the lines $x = x_0$ and $y = y_0$, or if Maple cannot verify the hyperbolic eigenfunction condition for the remainder r, the `prove` routine returns the value `FAIL` along with some additional information to indicate the cause of the failure.

It is very important to understand that in the derivation and proof of a particular identity, some or all of the actual arguments of the `derive` routine may be different than the corresponding actual arguments of the `prove` routine! The examples given below will clarify this point.

Here is the complete code for the `prove` routine:

```
prove := proc(expr::algebraic, eq1::name=algebraic,
              eq2::name=algebraic, n::posint)
  local x, x0, y, y0, r, Rx, Ry, i, zero, symmetric,
    Rxy, lambda;
  x, x0 := lhs(eq1), rhs(eq1);  # parse eq1
  y, y0 := lhs(eq2), rhs(eq2);  # parse eq2
  r := unapply(expr, x, y);  # final remainder
```

```
# Verify the multiplicity conditions for x = x0:
  Rx := unapply(Int(r(x,y), x), x, y);  # D[1](Rx) = r
  for i from 0 to n-1 do
    Rx := D[1](Rx);  # i-th order x-derivative
    zero := normal(Rx(x0,y));  # evaluated on x = x0
    if zero <> 0 then return FAIL, zero <> 0 end if;
  end do;
# Verify the multiplicity conditions for y = y0:
  symmetric := (r(x,y) = r(y,x)) and (x0 = y0);
  if not symmetric then  # verify this directly
  Ry := unapply(Int(r(x,y), y), x, y);  # D[2](Ry) = r
    for i from 0 to n-1 do
      Ry := D[2](Ry);  # i-th order y-derivative
      zero := normal(Ry(x,y0));  # evaluated on y = y0
      if zero <> 0 then return FAIL, zero <> 0 end if;
    end do;
  end if;
# Verify the hyperbolic eigenfunction condition for r:
  Rxy := D[2$n](D[1](Rx));  # 2n-th order mixed partial
  lambda := normal(Rxy(x,y)/r(x,y));  # the candidate
  if not type(lambda, constant)
    then return FAIL, 'lambda' = lambda  # non-constant
  end if;
  zero := normal(Rxy(x,y) - lambda*r(x,y));  # the test
  if zero <> 0 then return FAIL, zero <> 0 end if;
  true, 'lambda' = lambda  # proof complete!
end:
```

This routine uses function notation and derivative operators for clear and concise coding. The lower-order derivatives are saved and reused to compute higher-order derivatives more efficiently. Maple's `Int` function for *inert* integration is used to initialize the two differentiation loops so that the zero-order derivative can be recovered without coding an exceptional case.

If the remainder r is a symmetric function and the parameters x_0 and y_0 are identical, it suffices to verify the multiplicity conditions for the line $x = x_0$. The routine tests for this kind of symmetry—which occurs frequently in examples—thereby cutting the cost of verifying the multiplicity conditions in half.

Maple's `normal` function is used in precisely four places for two distinct purposes: to recognize zero and to cancel common factors in the numerator and denominator of a fraction. In the vast majority of cases, `normal` has nothing to do. For example, it is frequently the case that the normal derivatives of the remainder reduce to zero *immediately* upon evaluation, and it is also frequently the case that the fraction which calculates the candidate for the eigenvalue λ *immediately* reduces to a constant; in such cases, all of this happens *before* the `normal` function processes its argument. Since it usually has nothing to do, the `normal` function does not substantially increase the execution cost of the routine. At the same time, the `normal` function makes the routine more robust and easy to use by handling the few exceptional cases transparently.

A.2 Illustrative Examples

This section presents some examples which illustrate how the `derive` and `prove` routines can be used to derive and prove tensor product identities—automatically! These examples involve exponential and logarithm functions, trigonometric functions, and polynomial functions.

A.2.1 Exponential and Logarithm Functions

The first two examples illustrate the typical usage of this derivation and proof system. Later examples will present more exotic variations of these basic techniques.

We begin by deriving and proving the identity

$$\exp(x+y) \equiv \exp x \exp y.$$

We will define our own exponential function so that we can have greater control over the knowledge that Maple brings to bear during the course of our derivation and proof.

```
> define(Exp, Exp(0)=1, diff(Exp(x),x)=Exp(x));
> f := Exp(x+y);
```

$$f := \mathrm{Exp}(x+y)$$

```
> s := derive(f, x=0, y=0, 1);
```

$$s := \mathrm{Exp}(x)\,\mathrm{Exp}(y)$$

```
> prove(f-s, x=0, y=0, 1);
```

$$\mathit{true},\ \lambda = 1$$

```
> f=s;
```

$$\mathrm{Exp}(x+y) = \mathrm{Exp}(x)\,\mathrm{Exp}(y)$$

We will now derive and prove the inverse identity

$$\log(xy) \equiv \log x + \log y.$$

We will use Maple's built-in natural logarithm function in order to evaluate limits involving an infinite limit point. Note that the `derive` and `prove` routines are invoked with *essentially different arguments* in this example.

```
> f := log(x*y);
```

$$f := \ln(x\,y)$$

```
> s := derive(f, x=infinity, y=1, 2);
```

$$s := \ln(x) + \ln(y)$$

```
> prove(f-s, x=1, y=1, 1);
```

$$\mathit{true},\ \lambda = 0$$

```
> f=s;
```

$$\ln(x\,y) = \ln(x) + \ln(y)$$

A.2.2 Trigonometric Functions

We will now derive and prove the identity

$$\cos(x+y) \equiv \cos x \cos y - \sin x \sin y.$$

```
> define(Cos, Cos(0)=1, diff(Cos(x),x)=-Sin(x));
> define(Sin, Sin(0)=0, diff(Sin(x),x)=Cos(x));
> f := Cos(x+y):  s := derive(f, x=0, y=0, 2);
```

$$s := \mathrm{Cos}(x)\,\mathrm{Cos}(y) - \mathrm{Sin}(y)\,\mathrm{Sin}(x)$$

```
> prove(f-s, x=0, y=0, 2);
```

$$\mathit{true},\ \lambda = 1$$

```
> f=s;
```

$$\mathrm{Cos}(x+y) = \mathrm{Cos}(x)\,\mathrm{Cos}(y) - \mathrm{Sin}(y)\,\mathrm{Sin}(x)$$

Note that we can obtain the fundamental identity $\cos^2 x + \sin^2 x \equiv 1$ as a corollary of this tensor product identity by substituting $y = -x$ and using our knowledge that cosine and sine are even and odd functions, respectively.

Next, we will derive and prove the identity

$$\sin(x+y) \equiv \cos x \sin y + \sin x \cos y.$$

This example illustrates the need to expand the left-hand side of the identity in a weak dual asymptotic expansion at a *suitable* point (x_0, y_0). We will discover that the point $(0,0)$ used in the previous example is unsuitable for this example; however, we can use the point $(0, \frac{\pi}{2})$ instead.

```
> f := Sin(x+y):
> s := derive(f, x=0, y=0, 2);
```

$$s := FAIL$$

```
> definemore(Cos, Cos(Pi/2)=0);
> definemore(Sin, Sin(Pi/2)=1);
> s := derive(f, x=0, y=Pi/2, 2);
```

$$s := \mathrm{Sin}(x+\frac{\pi}{2})\,\mathrm{Sin}(y) - \mathrm{Cos}(y)\,\mathrm{Cos}(x+\frac{\pi}{2})$$

```
> prove(f-s, x=0, y=Pi/2, 2);
```

$$true,\ \lambda = 1$$

```
> f=s;
```

$$\mathrm{Sin}(x+y) = \mathrm{Sin}(x+\frac{\pi}{2})\,\mathrm{Sin}(y) - \mathrm{Cos}(y)\,\mathrm{Cos}(x+\frac{\pi}{2})$$

We can now use this intermediate identity to simplify itself by considering various special cases: Substituting $x = \frac{\pi}{2}$ and $y = 0$ into the intermediate identity shows that $\cos\pi = -1$. The fundamental identity $\cos^2 x + \sin^2 x \equiv 1$ further implies that $\sin\pi = 0$. Having established these special values, we can substitute $x = \frac{\pi}{2}$ into the intermediate identity to obtain the univariate identity $\sin(\frac{\pi}{2} + y) = \cos y$. Similarly, substituting $y = 0$ into the intermediate identity yields the univariate identity $\cos(x + \frac{\pi}{2}) = -\sin x$. These two univariate identities together reduce the intermediate identity to the standard form given above.

We will now derive and prove the same identity $\sin(x+y) \equiv \cos x \sin y + \sin x \cos y$ by a completely different technique: First, we will introduce a small nonzero symbolic parameter ε. Next, we will derive an intermediate identity which depends on ε and is valid for all ε in some deleted neighborhood of 0. Finally, we will eliminate this parameter by taking the limit of the intermediate identity as $\varepsilon \to 0$. This yields the desired identity quite readily.

Before we begin, let us induce amnesia by instructing Maple to forget everything we taught it about cosine and sine. This allows us to redefine these functions using the least knowledge possible.

```
> unassign(Cos, ‘diff/Cos‘, Sin, ‘diff/Sin‘);
> forget(diff);  forget(series);
> define(Cos, Cos(0)=1, diff(Cos(x),x)=-Sin(x));
> define(Sin, Sin(0)=0, diff(Sin(x),x)=Cos(x));
> f := Sin(x+y):  s := derive(f, x=0, y=epsilon, 2);
```

$$s := \frac{\mathrm{Sin}(y)\,\mathrm{Sin}(x+\varepsilon)}{\mathrm{Sin}(\varepsilon)} + (\mathrm{Sin}(\varepsilon)\,\mathrm{Cos}(x+\varepsilon) - \mathrm{Cos}(\varepsilon)\,\mathrm{Sin}(x+\varepsilon))$$
$$(-\mathrm{Cos}(y)\,\mathrm{Sin}(\varepsilon) + \mathrm{Sin}(y)\,\mathrm{Cos}(\varepsilon))/(\mathrm{Sin}(\varepsilon)\,(\mathrm{Sin}(\varepsilon)^2 + \mathrm{Cos}(\varepsilon)^2))$$

```
> prove(f-s, x=0, y=epsilon, 2);
```

$$\mathit{true},\ \lambda = 1$$

```
> limit(f=s, epsilon=0);
```

$$\mathrm{Sin}(x+y) = \mathrm{Cos}(x)\,\mathrm{Sin}(y) + \mathrm{Cos}(y)\,\mathrm{Sin}(x)$$

We will now derive and prove the tensor product identities for $\cos(x+y)$ and $\sin(x+y)$ *simultaneously* by introducing a new independent variable z. This will yield an intermediate identity which is valid for all real x, y, and z. Since the intermediate identity will consist of first-degree polynomials in z on both the left- and right-hand sides, we can equate the coefficients of like powers of z to obtain two identities which are valid for all real x and y.

```
> f := Cos(x+y) + z*Sin(x+y):
> s := derive(f, x=0, y=0, 2);
```

$$s := (\mathrm{Cos}(y) + z\,\mathrm{Sin}(y))\,(\mathrm{Sin}(x)\,z + \mathrm{Cos}(x)) - \mathrm{Sin}(y)\,\mathrm{Sin}(x)\,(z^2+1)$$

```
> s := sort(collect(s, z));
```

$$s := -\mathrm{Sin}(y)\,\mathrm{Sin}(x) + \mathrm{Cos}(x)\,\mathrm{Cos}(y) + (\mathrm{Sin}(y)\,\mathrm{Cos}(x) + \mathrm{Sin}(x)\,\mathrm{Cos}(y))\,z$$

```
> prove(f-s, x=0, y=0, 2);
```

$$\mathit{true},\ \lambda = 1$$

```
> map(coeff, f=s, z, 0);
```

$$\mathrm{Cos}(x+y) = \mathrm{Cos}(x)\,\mathrm{Cos}(y) - \mathrm{Sin}(y)\,\mathrm{Sin}(x)$$

```
> map(coeff, f=s, z, 1);
```

$$\mathrm{Sin}(x+y) = \mathrm{Sin}(y)\,\mathrm{Cos}(x) + \mathrm{Sin}(x)\,\mathrm{Cos}(y)$$

A.2.3 Polynomial Functions

We will now derive and prove this instance of the binomial theorem:

$$(x+y)^4 \equiv x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

```
> f := (x+y)^4:  s := derive(f, x=infinity, y=0, 5):
> s := sort(s, [x,y]):  prove(f-s, x=0, y=0, 2);
```

$$\mathit{true},\ \lambda = 0$$

```
> f=s;
```

$$(x+y)^4 = x^4 + 4\,x^3\,y + 6\,x^2\,y^2 + 4\,x\,y^3 + y^4$$

We will now derive and prove this instance of the discrete convolution formula for monomials:

$$(x^5 - y^5)/(x-y) \equiv x^4 + x^3y + x^2y^2 + xy^3 + y^4.$$

To facilitate the proof step, we will multiply the remainder by $x - y$ to remove the line of singularities $y = x$ from the left-hand side of the identity. After establishing the intermediate identity $(x^5 - y^5) \equiv (x^4 + x^3y + x^2y^2 + xy^3 + y^4)(x-y)$ for all real x and y, the desired identity follows by dividing by $x - y$ for $y \neq x$. Of course, expanding the right-hand side into a telescoping sum is the *simplest* way to prove the intermediate identity; however, our purpose here is *not* to give the simplest possible proof, but rather to demonstrate that this identity fits nicely into our conceptual framework.

```
> F := x^5 - y^5:  f := F/(x-y):
> s := derive(f, x=infinity, y=0, 5):  s := sort(s, [x,y]):
> S := (x-y)*s:  prove(F-S, x=0, y=0, 3);
```

$$true, \ \lambda = 0$$

```
> f=s;
```

$$\frac{x^5 - y^5}{x - y} = x^4 + x^3\, y + x^2\, y^2 + x\, y^3 + y^4$$

For many more examples which demonstrate the widespread applicability of these automated derivation and proof techniques, please see the author's recent ISSAC conference paper [Cha03].

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